Control Architecture for Synchronization and Tracking in Descriptor Multi-Agent Systems

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***Abstract*— A control architecture is proposed for a descriptor multi-agent system to achieve the following objectives in the closed loop: i) the state response is impulse-free, and ii) the states of follower agents synchronize with the states of leader agent satisfying a pre-specified *minimum decay rate* (helps to regulate the synchronization speed). By performing a sequence of system transformations through orthogonal matrices, and using a *parametric robust control* framework, the synchronization problem is addressed, where the associated gain parameters are computed by solving a linear matrix constraint problem, and using the largest eigenvalue of network Laplacian matrix. The implementation of some feedforward gain matrices ensures that the constant reference tracking objective is also achieved, in addition to the synchronization. The proposed algorithm is demonstrated with a numerical example.**

***Index Terms*— Descriptor systems, multi-agent systems, distributed control, synchronization.**

I. Introduction

A multi-agent system (MAS) consists of a group of dynamical systems (*agents*), which work cooperatively via appropriate communication links to achieve some collective objectives, such as: i) consensus, ii) leader-follower synchronization, iii) formation, and iv) rendezvous (see [1]–[5] and the references therein). A MAS is referred to as *descriptor multi-agent system* (DMAS), when the individual agents are descriptor systems (DSs). Often the physical systems with *state constraints* are suitably modeled as DSs. The examples of DSs are: power networks [6], overhead trolley crane [7], robotic manipulators [8], biological systems [9] and cyber-physical systems [10]. Some of the characteristics, which make DSs different from ordinary state-space systems, are: the state response of a DS may not exist if it is not *regular*, and could be *impulsive* for inconsistent initial conditions and/or non-smoothness of (control) input [8], [11], [12]. The presence of impulse in the system response may damage or saturate the physical components [11], and hence, should be eliminated.

In this work, we consider a DMAS, where one of the agents is a leader and others are follower. Then, we design a feedback control to achieve the following objectives in the closed loop: i) *impulse-free* response, ii) leader-follower synchronization, that is, the states of follower agents converge to the leader state, satisfying a *specified decay rate* (helps to regulate the time required for synchronization; referred to it as synchronization speed), and iii) reference tracking, where the states of follower agents first converge to the leader state, satisfying a specified decay rate, and then they all converge to the external reference trajectory, which is accessible only to the leader. To achieve these objectives, we propose a combination of *decentralized* (using local state measurements) and *distributed* (using neighborhood state measurements) feedback control architecture. Our contributions to design these control laws are as follows. We first design a feedback gain matrix for decentralized control law to make the individual agents impulse-free and place their finite poles within a stability region Ω_l (left half of a vertical line) in the complex plane. Then, to achieve synchronization (with flexibility of regulating synchronization speed), we use distributed feedback control, where we prove that this objective, including impulse-freeness, can be achieved by designing a feedback gain matrix $\bar{K}$ and some scalar gains that assign finite poles of an *error dynamics* within a stability region $\Omega_f \subset \Omega_l$. Using a sequence of transformations through orthogonal matrices, we show that assigning the finite poles of error dynamics is equivalent to the eigenvalues assignment of a set of ordinary state-space systems, which is then, translated into the eigenvalues assignment of *a single* ordinary state-space system with the help of *parametric robust control* framework. Finally, a linear matrix constraint (LMC) feasibility problem is formulated to compute $\bar{K}$ and the scalar gains. By designing some feedforward gain matrices, we achieve constant reference tracking objective.

The existing literature on DMAS consider consensus as a collective control objective [13]–[17]. For this, relative outputs based static and dynamic feedback control [13], [15], a combination of event-triggered and impulsive hybrid control [17] and external disturbance suppressing control algorithms [14], [18] are proposed in the literature. Moreover, the problem of *bipartite consensus* and *bipartite containment consensus* are considered in [16] and [19], respectively. In the existing work, it is an underlying assumption that the agents are impulse-free, which in turn helps to use the generalized Riccati equation (for descriptor system) to design local controllers for the agents. In contrast, our approach does not make such an assumption. Moreover, with our approach, one can modify the speed of synchronization, which is crucial in many practical applications. Since we have used only orthogonal matrices, our algorithm is numerically efficient.

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Notations: $\otimes$: Kronecker product, $\det(A)$: determinant of matrix A, $\deg\{\cdot\}$: degree of a polynomial, $\text{diag}\{\cdot\}$: diagonal matrix, I_r: $r\times r$ identity matrix, $\lambda_{max}(A)$ and $\lambda_{min}(A)$ denote maximum and minimum eigenvalues of A, $||\cdot||_2$: 2-norm, and $A \succ 0$ ($A \prec 0$) denotes that A is symmetric positive (negative) definite matrix.

II. Problem Formulation and Main Results

A. Problem Formulation

We consider a network of $r+1$ number of identical agents, which form a DMAS, where the individual agent's dynamics is represented by a descriptor system as follows [8], [11]:

$$E\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \text{ for } i = 0, 1, 2, \cdots, r. \quad (1)$$

In (1), $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^p$ are the state and input vectors, respectively of i^{th} agent. Out of these $r+1$ agents, we consider one of the agents as *leader* and the remaining r agents as *follower*. Without loss of generality, the leader dynamics is represented by (1) for $i = 0$, and the followers dynamics are represented by (1) for $i = 1, 2, \cdots, r$. The considered DMAS is represented by an undirected connected graph $G(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \{v_0, v_1, \cdots, v_r\}$ is the set of vertices and $\mathcal{E}$ is the set of edges. In $G(\mathcal{V}, \mathcal{E})$, as shown in Figure 1(a), a vertex v_i represents the i^{th} agent in the network, and an edge between the vertices v_i and v_j represents the communication link between the agents i and j. Let a subgraph $G_s(\mathcal{V}', \mathcal{E}')$ be constructed from $G(\mathcal{V}, \mathcal{E})$ by removing v_0 and the edges, which are connected to v_0, as shown in Figure 1(b). Two vertices v_i and v_j in $G_s(\mathcal{V}', \mathcal{E}')$ are said to be *adjacent*, if there is an edge between v_i and v_j. We assign an *adjacency matrix* $\mathcal{A}(G_s) \in \mathbb{R}^{r\times r}$ for $G_s(\mathcal{V}', \mathcal{E}')$, where its $(i, j)^{th}$ element is $a_{ij} > 0$, whenever the vertices v_i and v_j are adjacent in $G_s(\mathcal{V}', \mathcal{E}')$, otherwise it is zero. It is assumed that the edge weights a_{ij} satisfy the following conditions: i) $a_{ij} = a_{ji}$ and $a_{ii} = 0$ (no self-loop). We define the *degree matrix* $\mathcal{D}(G_s)$ of $G_s(\mathcal{V}', \mathcal{E}')$ as follows: $\mathcal{D}(G_s) = \text{diag}\{d_1, d_2, \cdots, d_r\}$, where $d_i := \sum_{j=1}^{r} a_{ij}$ for $j \neq i$, is the degree of vertex v_i (or vertex degree). Further, the *Laplacian matrix* $\mathcal{L}(G_s)$ of $G_s(\mathcal{V}', \mathcal{E}')$ is defined as follows: $\mathcal{L}(G_s) := \mathcal{D}(G_s) - \mathcal{A}(G_s)$. With this setup, in this work, we address the following problems.

Problem 1: For a given DMAS, with agent dynamics as in (1), design control protocols $u_i(t)$ for the agents such that: i) the closed loop system is impulse-free, ii) $x_0(t)$ converges to zero, satisfying a minimum decay rate α_1 and iii) follower states $x_i(t)$ synchronize with (or converge to) the leader state $x_0(t)$, that is, $\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0$, for $i = 1, 2, \cdots, r$, satisfying a minimum decay rate $\alpha_2 > \alpha_1$.

Problem 2: Let the leader agent in a DMAS has access to a constant external reference signal $\bar{r}(t) = [\gamma_1\ \gamma_2\ \cdots\ \gamma_n]^T$, where $\gamma_i \in \mathbb{R}$ for $t \geq 0$. Then, design control protocols $u_i(t)$ for the agents such that objectives i) and iii) of Problem 1 are satisfied, and $\lim_{t\to\infty}(x_0(t) - \bar{r}(t)) = 0$, satisfying a minimum decay rate α_1 (constant reference tracking).

B. Controller Synthesis for Synchronization

Consider the following assumptions on (1): i) E is singular, and $\text{rank}(E) = n_0 < n$, ii) pair (E, A) is regular, and iii) it is C-controllable [8, Chapter 4]. Defining the sets: $\mathcal{M}_0 := \{0, 1, \cdots, r\}$ and $\mathcal{M}_1 := \{1, 2, \cdots, r\}$, we propose the following control architecture to address Problem 1:

$$u_i(t) = \widetilde{u}_i(t) + \bar{u}_i(t), \text{ for } i \in \mathcal{M}_0, \quad (2)$$

where the control signals $\widetilde{u}_i(t)$ and $\bar{u}_i(t)$ are defined as:

$$\widetilde{u}_i(t) = Kx_i(t), \text{ for all } i \in \mathcal{M}_0, \quad (3a)$$

$$\bar{u}_0(t) = 0, \ \bar{u}_i(t) = \bar{K}\bar{z}_i(t), \text{ for } i \in \mathcal{M}_1. \quad (3b)$$

In (3a) and (3b), $K \in \mathbb{R}^{p\times n}$ and $\bar{K} \in \mathbb{R}^{p\times n}$ are the feedback gain matrices, and the signal $\bar{z}_i(t)$ in (3b) is defined as follows for all $i \in \mathcal{M}_1$:

$$\bar{z}_i(t) = \mu\left(\sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j)\right) + \beta_i\left(g_i(x_i - x_0)\right), \quad (4)$$

where μ and β_i are some positive real numbers, whose choice will be discussed later, and $\mathcal{N}_i$ is the set of neighborhood agents of i^{th} agent, which is defined as follows:

$$\mathcal{N}_i := \{j \mid \text{vertices } v_i \text{ and } v_j \text{ are adjacent in } G_s(\mathcal{V}', \mathcal{E}')\}.$$

According to control laws (3a) and (3b), the leader agent uses only its local state information, while the followers rely on information from their neighboring agents also. Moreover, it is important to note that although the leader agent is not directly connected to all the agents, the follower agents can get access of the leader state x_0 from their neighborhood agents, as shown in Figure 1(a). This is possible, since we have assumed that the network graph $G(\mathcal{V}, \mathcal{E})$ is connected. Notice that we have assigned some weights ($a_{ij} > 0$) to each edge of $G_s(\mathcal{V}', \mathcal{E}')$, which can be considered as the *cost* (energy utilization, uncertainties and constraints on communication channels) associated with transferring the state information from one agent to its neighbor. Analogously, we consider the quantity g_i in (4) as the *weight* associated with transferring the leader state x_0 to the i^{th} follower agent through neighborhood agents. The quantity $g_i > 0$ is not a design parameter, rather a given quantity, similar to the edge weight a_{ij}. Without loss of generality, one can set: $a_{ij} = g_i = 1$, when there is no cost associated with state transformations. The design parameters in the control law (2) are gain matrices: K, $\bar{K}$, and scalar gains: μ, β_i.

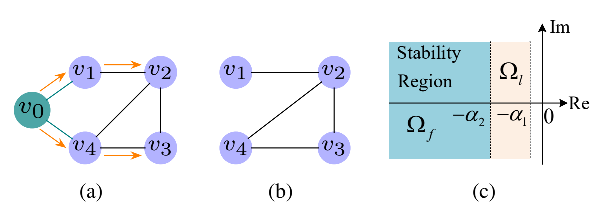


Fig. 1. (a) Graph $G(\mathcal{V}, \mathcal{E})$, representing a DMAS with five agents, where the arrows represent flow of leader state x_0 and external reference $\bar{r}$ (only accessible to leader) to the follower agents. (b) A sub-graph of $G(\mathcal{V}, \mathcal{E})$. (c) Stability regions Ω_l and Ω_f in the complex plane.

Using the control protocol (3a), the closed-loop dynamics of i^{th} agent becomes:

$$E\dot{x}_i(t) = (A + BK)x_i(t) + B\bar{u}_i(t), \text{ for } i \in \mathcal{M}_0. \quad (5)$$

We propose to design the feedback gain matrix K according to the procedure given in [20] such that each agent in (5): i) becomes impulse-free, ii) and its finite poles are placed within a stability region $\Omega_l := \{s \in \mathbb{C} \mid \text{Real}(s) < -\alpha_1\}$, as shown in Figure 1(c), where $\alpha_1 > 0$. The stability region Ω_l is chosen to ensure that the individual agents are stable and/or their state trajectories converge to zero satisfying a minimum decay rate of α_1. Then, applying the control law (3b) in (5), and defining $A_K := A + BK$, we have:

$$E\dot{x}_0(t) = A_K x_0(t), \qquad \text{for } i = 0, \tag{6a}$$
$$E\dot{x}_i(t) = A_K x_i(t) + B\bar{K}\bar{z}_i(t), \quad \text{for } i \in \mathcal{M}_1. \tag{6b}$$

Define the following vectors: $\mathbf{x} := [x_1^T \; x_2^T \; \cdots \; x_r^T]^T$, $\mathbf{x}_0 := [x_0^T \; x_0^T \; \cdots \; x_0^T]^T$ and $\bar{\mathbf{z}} := [\bar{z}_1^T \; \bar{z}_2^T \; \cdots \; \bar{z}_r^T]^T$. Further, define a matrix $\mathcal{L}_g$ as follows:

$$\mathcal{L}_g := \mu\mathcal{L}(G_s) + \text{diag}\{\beta_1 g_1, \beta_2 g_2, \cdots, \beta_r g_r\}, \tag{7}$$

where $\mathcal{L}(G_s)$ is the Laplacian matrix of network (sub)graph $G_s(\mathcal{V}', \mathcal{E}')$. Then, the relations in (4) can compactly be represented as follows:

$$\bar{\mathbf{z}} := (\mathcal{L}_g \otimes I_n)(\mathbf{x} - \mathbf{x}_0). \tag{8}$$

Further, the dynamics of the leader and follower agents in (6), using the relation (8), can be represented as:

$$(I_r \otimes E)\dot{\mathbf{x}}_0 = (I_r \otimes A_K)\mathbf{x}_0, \tag{9a}$$
$$\begin{aligned}(I_r \otimes E)\dot{\mathbf{x}} = &\left((I_r \otimes A_K) + (\mathcal{L}_g \otimes B\bar{K})\right)\mathbf{x} \\ &- (\mathcal{L}_g \otimes B\bar{K})\mathbf{x}_0.\end{aligned} \tag{9b}$$

Define a vector $\widetilde{\mathbf{x}}(t)$ as follows: $\widetilde{\mathbf{x}}(t) = [\mathbf{x}_0^T \;\; \mathbf{x}^T]^T$. Then, the relations in (9b) and (9a) can be written as follows, which is the overall dynamics of closed-loop DMAS:

$$\mathbf{E}_{cl}\dot{\widetilde{\mathbf{x}}}(t) = \mathbf{A}_{cl}\widetilde{\mathbf{x}}(t), \tag{10}$$

where the matrices $\mathbf{E}_{cl}$ and $\mathbf{A}_{cl}$ are:

$$\mathbf{E}_{cl} = \begin{bmatrix} I_r\otimes E & 0 \\ 0 & I_r\otimes E \end{bmatrix}, \mathbf{A}_{cl} = \begin{bmatrix} I_r\otimes A_K & 0 \\ -\mathcal{L}_g\otimes B\bar{K} & (I_r\otimes A_K)+(\mathcal{L}_g\otimes B\bar{K}) \end{bmatrix}.$$

Now, define an error signal as follows: $\delta_i(t) := x_i(t) - x_0(t)$ for $t \geq 0$ and $i \in \mathcal{M}_1$. Then, by defining an error vector: $\delta = [\delta_1^T \; \delta_2^T \; \cdots \; \delta_r^T]^T$, we have: $\delta = \mathbf{x} - \mathbf{x}_0$. By taking derivative on both sides of $\delta = \mathbf{x} - \mathbf{x}_0$ and pre-multiplying $(I_r \otimes E)$, following relation is obtained:

$$(I_r \otimes E)\dot{\delta} = (I_r \otimes E)\dot{\mathbf{x}} - (I_r \otimes E)\dot{\mathbf{x}}_0, \tag{11}$$

which yields the following *error dynamics* using (9):

$$(I_r \otimes E)\dot{\delta} = \left((I_r \otimes A_K) + (\mathcal{L}_g \otimes B\bar{K})\right)\delta. \tag{12}$$

Let $E_\delta := (I_r \otimes E)$ and $A_\delta := (I_r \otimes A_K) + \left(\mathcal{L}_g \otimes B\bar{K}\right)$. Then, the following result holds.

Proposition 1: Let the matrix $\mathcal{L}_g$ be as in (7), and $\bar{\lambda}_i$, which is a function of μ and β_i, is an eigenvalue of $\mathcal{L}_g$. Then, $\Lambda(E_\delta, A_\delta) = \bigcup_{i=1}^{r} \Lambda(E, A_K + \bar{\lambda}_i B\bar{K})$, where $\Lambda(\cdot)$ refers to the set of all finite poles.

Proof: Since the matrix $\mathcal{L}_g$ is symmetric, one can find an orthogonal matrix Q ($Q^T Q = I_r$) such that $Q^T\mathcal{L}_g Q = D_\Lambda$, where $D_\Lambda = \text{diag}\{\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_r\}$ [21]. Now define a new variable as follows: $\epsilon(t) := (Q^T \otimes I_n)\delta(t)$. Then, it follows from the error dynamics (12) that:

$$(I_r \otimes E)(Q \otimes I_n)\dot{\epsilon}(t) = A_\delta(Q \otimes I_n)\epsilon(t). \tag{13}$$

Furthermore, since $(I_r \otimes E)(Q \otimes I_n) = (Q \otimes I_n)(I_r \otimes E)$, the relation (13) produces:

$$(I_r \otimes E)\dot{\epsilon}(t) = \left[(I_r \otimes A_K) + (D_\Lambda \otimes B\bar{K})\right]\epsilon(t). \tag{14}$$

The error dynamics (12) and the transformed descriptor system (14) are equivalent, and hence, they have the same set of finite poles. Moreover, (14) can be represented as:

$$E\dot{\epsilon}_i(t) = (A_K + \bar{\lambda}_i B\bar{K})\epsilon_i(t), \qquad \text{for } i \in \mathcal{M}_1, \tag{15}$$

and hence, the proposition holds. ■

Theorem 1: Assume that the feedback gain matrix $\bar{K}$, and scalars μ, β_i are designed such that the finite poles of the error dynamics (12) are confined within the stability region: $\Omega_f := \{s \in \mathbb{C} \mid \text{Real}(s) < -\alpha_2, \; \alpha_1 < \alpha_2\}$, as shown in Figure 1(c). Then the following statements hold.

1) The follower states $x_i(t)$ will converge to the leader state $x_0(t)$ satisfying a minimum decay rate α_2.
2) The closed loop state trajectory $\widetilde{\mathbf{x}}$ in (10) is free from impulses (and its derivatives, in distribution sense).

Proof: *Proof of first statement:* Since the feedback gain matrix $\bar{K}$, and scalars μ, β_i are designed such that the finite poles of (12) cluster within the stability region: Ω_f, it follows from [8, Chapter 3] and [22] that $\|\delta(t)\|_2 \leq \phi e^{-\alpha_2 t}\|\delta(0)\|_2$, where ϕ is a constant and α_2 is the decay rate. Hence, $\lim_{t\to\infty}\|\delta(t)\|_2 = 0$. Since $\delta = \mathbf{x} - \mathbf{x}_0$, we have: $\lim_{t\to\infty}\|(\mathbf{x}(t) - \mathbf{x}_0(t))\|_2 = 0$, and hence, for $i \in \mathcal{M}_1$, $\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0$, and the first statement hold.

Proof of second statement: It follows from [8, Theorem 7.1] that the state trajectory $\tilde{\mathbf{x}}$ is free from impulses if and only if: $\deg\{\det(s\mathbf{E}_{cl} - \mathbf{A}_{cl})\} = \text{rank}(\mathbf{E}_{cl})$. Using the definition of $\mathbf{E}_{cl}$ and $\mathbf{A}_{cl}$ as in (10), we have:

$$\deg\{\det(s\mathbf{E}_{cl} - \mathbf{A}_{cl})\} = \deg\left\{\det\left(\begin{bmatrix} I_r\otimes sE & 0 \\ 0 & I_r\otimes sE \end{bmatrix} - \begin{bmatrix} I_r\otimes A_K & 0 \\ -\mathcal{L}_g\otimes B\bar{K} & (I_r\otimes A_K)+(\mathcal{L}_g\otimes B\bar{K}) \end{bmatrix}\right)\right\}.$$

Consider the orthogonal matrix Q such that $Q^T\mathcal{L}_g Q = D_\Lambda$ (refer to the proof of Proposition 1), and define a new matrix: $\bar{Q} = \begin{bmatrix} Q\otimes I_n & 0 \\ 0 & Q\otimes I_n \end{bmatrix}$. Then, we have:

$$\begin{aligned}
&\deg\{\det(s\mathbf{E}_{cl} - \mathbf{A}_{cl})\} \\
&= \deg\left\{\det\left(\bar{Q}^T \begin{bmatrix} I_r\otimes sE & 0 \\ 0 & I_r\otimes sE \end{bmatrix}\bar{Q} - \right.\right. \\
&\qquad \left.\left.\bar{Q}^T\begin{bmatrix} I_r\otimes A_K & 0 \\ -\mathcal{L}_g\otimes B\bar{K} & (I_r\otimes A_K)+(\mathcal{L}_g\otimes B\bar{K}) \end{bmatrix}\bar{Q}\right)\right\} \\
&= \deg\left\{\det\left(\begin{bmatrix} I_r\otimes(sE-A_K) & 0 \\ D_\Lambda\otimes B\bar{K} & I_r\otimes(sE-A_K)-(D_\Lambda\otimes B\bar{K}) \end{bmatrix}\right)\right\} \\
&= \deg\left\{\det\left(I_r \otimes (sE - A_K)\right)\right\} + \\
&\qquad \deg\left\{\det\left((I_r \otimes (sE - A_K)) - (D_\Lambda \otimes B\bar{K})\right)\right\}.
\end{aligned} \tag{16}$$

Define the following quantities: $\kappa_1 := \deg\{\det(sE - A_K)\}$ and $\kappa_2 := \deg\{\det\left(sE - A_K - \bar{\lambda}_i B\bar{K}\right)\}$. Then, from (16), we have the following relation:

$$\deg\{\det(s\mathbf{E}_{cl} - \mathbf{A}_{cl})\} = r(\kappa_1 + \kappa_2). \tag{17}$$

Since the gain matrix K is designed such that the pair (E, A_K) is impulse-free, it follows from [8, Theorem 7.1] that $\kappa_1 = \deg\{\det(sE - A_K)\} = n_0$, where $n_0 = \text{rank}(E)$. Moreover, we show in the forthcoming result (see Lemma 1) that $\bar{K}$, μ and β_i are designed in such a way that n_0 number of finite poles of each pair $(E, A_K + \bar{\lambda}_i B\bar{K})$ for $i \in \mathcal{M}_1$ are placed within the stability region Ω_f. Hence, it follows that $\kappa_2 = n_0$. Then, we have: $\deg\{\det(s\mathbf{E}_{cl} - \mathbf{A}_{cl})\} = 2rn_0 = \text{rank}(\mathbf{E}_{cl})$, and hence, the second statement holds. ∎

According to Theorem 1, since $x_i(t)$ converges to $x_0(t)$ satisfying a minimum decay rate α_2, the quantity α_2 determines the synchronization speed in a DMAS. Hence, one can always modify the synchronization speed by choosing α_2 appropriately, and then designing $\bar{K}$, β_i, μ accordingly. The design of $\bar{K}$, β_i and μ are mentioned in Theorem 2. We have chosen $\alpha_2 > \alpha_1$ for Ω_f to ensure that the time required for synchronization is less than the time required for leader state to converge to zero. With the chosen Ω_l and Ω_f, the follower states $x_i(t)$ first converge to leader state $x_0(t)$, and then, they all follow $x_0(t)$, which finally converges to zero satisfying a minimum decay rate α_1.

Consider an orthogonal matrix Z such that the matrices associated with the dynamical systems (obtained from (15)): $ZE\dot{\epsilon}_i(t) = (ZA_K + \bar{\lambda}_i ZB\bar{K})\epsilon_i(t)$, for $i \in \mathcal{M}_1$, have the following form (one may use singular value decomposition (SVD) of E to obtain Z): $ZE = \left[\begin{smallmatrix} E_1 \\ 0 \end{smallmatrix}\right]$, $ZA_K = \left[\begin{smallmatrix} A_{K_1} \\ A_{K_2} \end{smallmatrix}\right]$, $ZB = \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right]$, where $E_1 \in \mathbb{R}^{n_0 \times n}$, $A_{K_1} \in \mathbb{R}^{n_0 \times n}$, $A_{K_2} \in \mathbb{R}^{(n-n_0)\times n}$, $B_1 \in \mathbb{R}^{n_0 \times p}$ and $B_2 \in \mathbb{R}^{(n-n_0)\times p}$. Since the pair (E, A_K) is impulse-free, the orthogonal matrix Z exists such that E_1 and A_{K_2} are of full row rank. Now, consider another two orthogonal matrices V and W such that the matrix $ZE = V\left[\begin{smallmatrix} \Sigma_{n_0} & 0 \\ 0 & 0 \end{smallmatrix}\right]W^T$. Let the orthogonal matrices V and W be partitioned as: $V = \left[\begin{smallmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{smallmatrix}\right]$ and $W = \left[\begin{smallmatrix} W_1 & W_2 \end{smallmatrix}\right]$, respectively, where $V_{11} \in \mathbb{R}^{n_0 \times n_0}$, $V_{12} \in \mathbb{R}^{n_0 \times (n-n_0)}$, $V_{21} \in \mathbb{R}^{(n-n_0)\times n_0}$, $V_{22} \in \mathbb{R}^{(n-n_0)\times(n-n_0)}$, and $W_1 \in \mathbb{R}^{n \times n_0}$, $W_2 \in \mathbb{R}^{n \times (n-n_0)}$. Using these partitioned matrices, we further define the following matrices: $A_{d_{11}} := \Sigma_{n_0}^{-1} V_{11}^T \left(A_{K_1}W_1 - A_{K_1}W_2(A_{K_2}W_2)^{-1}A_{K_2}W_1\right)$, $B_{d_1} := \Sigma_{n_0}^{-1} V_{11}^T \left(B_1 - A_{K_1}W_2(A_{K_2}W_2)^{-1}B_2\right)$, and $\bar{K} := \left[\bar{K}_1 \quad 0\right] W^T$, where $A_{d_{11}} \in \mathbb{R}^{n_0 \times n_0}$, $B_{d_1} \in \mathbb{R}^{n_0 \times p}$ and $\bar{K}_1 \in \mathbb{R}^{p \times n_0}$. Then, the following result holds.

Lemma 1: The union of the finite poles of a set of descriptor systems (15) is equal to the union of the eigenvalues of a set of matrices: $A_{d_{11}} + \bar{\lambda}_i B_{d_1}\bar{K}_1$, for all $i \in \mathcal{M}_1$.

Proof: The proof directly follows from the result [20, Theorem 3], and hence, is skipped here. ∎

Theorem 2: For a given stability region Ω_f, let the matrix $\bar{K}_1$ and symmetric positive definite (SPD) matrices: X and P be the solution of matrix equation:

$$A_{d_{11}}X + XA_{d_{11}}^T + N + 2\alpha_2 X = -P, \tag{18}$$

where $N := B_{d_1}\bar{K}_1 X + X\bar{K}_1^T B_{d_1}^T$. Using the solution X, P and $\bar{K}_1$ of (18), let the scalars β_i and μ are chosen as:

$$\beta_i = \frac{1}{g_i},\ \forall\ i \in \mathcal{M}_1;\quad 0 < \mu < \frac{(\lambda_{min}(P)/||N||_2)}{\lambda_{max}(\mathcal{L}(G_s))}. \tag{19}$$

Then, by denoting $\Lambda(\cdot)$ as the set of eigenvalues of a matrix, the following set inclusion relation holds:

$$\bigcup_{i=1}^{r} \Lambda\left(A_{d_{11}} + \bar{\lambda}_i B_{d_1}\bar{K}_1\right) \subset \Omega_f. \tag{20}$$

Proof: Since the matrix equation (18) holds, it directly follows from [22] that the eigenvalues of $\left(A_{d_{11}} + B_{d_1}\bar{K}_1\right)$ are clustered within the stability region Ω_f. We now use a *parametric robust control* framework [23, Chapter 12] to prove the result. Let $\theta \in \mathbb{R}$ be an uncertain parameter. Assume that the feedback gain matrix $\bar{K}_1$ is subjected to the additive parametric perturbation, which is represented as follows: $\bar{K}_1(\theta) := \bar{K}_1 + \theta\bar{K}_1$. Then, we have:

$$A_{d_{11}} + B_{d_1}\bar{K}_1(\theta) = A_{d_{11}} + B_{d_1}\bar{K}_1 + \theta B_{d_1}\bar{K}_1. \tag{21}$$

For every $\bar{K}_1$, which is a solution of the matrix equation (18), following relation holds: $\Lambda\left(A_{d_{11}} + B_{d_1}\bar{K}_1\right) \subset \Omega_f$. We are now interested in finding a range $\mathcal{R}$ for the uncertain parameter θ such that the following set relation holds:

$$\Lambda\left(A_{d_{11}} + B_{d_1}\bar{K}_1 + \theta B_{d_1}\bar{K}_1\right) \subset \Omega_f,\ \text{for all}\ \theta \in \mathcal{R}. \tag{22}$$

It follows from [22] that the relation (22) holds if and only if the following matrix inequality holds for every $\theta \in \mathcal{R}$:

$$\begin{aligned}&\left(A_{d_{11}} + B_{d_1}\bar{K}_1 + \theta B_{d_1}\bar{K}_1\right)X + \\ &\quad X\left(A_{d_{11}} + B_{d_1}\bar{K}_1 + \theta B_{d_1}\bar{K}_1\right)^T + 2\alpha_2 X \prec 0.\end{aligned} \tag{23}$$

Since $\bar{K}_1$ and X satisfy the matrix equation (18), following equivalent matrix inequality is obtained from (23):

$$-P + \theta\left(B_{d_1}\bar{K}_1 X + X\bar{K}_1^T B_{d_1}^T\right) \prec 0. \tag{24}$$

Further, using the definition of N, the matrix inequality (24) can be written as: $P - \theta N \succ 0$. It then follows from the Weyl's result [21, Theorem 4.3.1] that the relation: $P - \theta N \succ 0$ holds, if the following inequality holds:

$$\lambda_{max}(\theta N) < \lambda_{min}(P). \tag{25}$$

Since θN is a symmetric matrix, $|\lambda_i(\theta N)| = \sigma_i(\theta N)$, where $\lambda_i(\cdot)$ and $\sigma_i(\cdot)$ refer to the i^{th} eigenvalue and singular value, respectively, and $|\cdot|$ refers to the absolute value. Hence, for $i = 1, 2, \cdots, n_0$, following relation holds:

$$\max_i |\lambda_i(\theta N)| = \sigma_{max}(\theta N) = \|\theta N\|_2 = |\theta|\|N\|_2, \tag{26}$$

where $\sigma_{max}(\cdot)$ refers to the largest singular value of a matrix. Further, since $\lambda_{max}(\theta N) \leq \max_i |\lambda_i(\theta N)|$, it follows from the relations (26) that the inequality (25) holds if:

$$|\theta|\|N\|_2 < \lambda_{min}(P)\ \equiv\ |\theta| < \frac{\lambda_{min}(P)}{\|N\|_2}. \tag{27}$$

Therefore, every θ satisfying (27) ensures that the matrix inequality (23) holds, and hence, the set inclusion relation (22) holds. Assuming that θ is a non-negative real number, the inequality in (27) can be replaced with $\theta < \frac{\lambda_{min}(P)}{\|N\|_2}$. Hence, by setting $\mathcal{R} = \left[0, \frac{\lambda_{min}(P)}{\|N\|_2}\right)$, the set inclusion relation (22) holds for all $\theta \in \mathcal{R}$.

Assume that the eigenvalues of $\mathcal{L}_g$ and $\mathcal{L}(G_s)$ are arranged in decreasing (or increasing) order. Then, according to the definition of $\mathcal{L}_g$, as in (7), it follows that for the choice of β_i, as in (19), the eigenvalues of $\mathcal{L}_g$ are: $\bar{\lambda}_i = 1+\mu\lambda_i$, where λ_i, for $i \in \mathcal{M}_1$, is an eigenvalue of $\mathcal{L}(G_s)$. Further, since μ satisfies (19), we have: $\mu\lambda_{max}(\mathcal{L}(G_s)) < \frac{\lambda_{min}(P)}{\|N\|_2}$. In addition, since $\mu\lambda_i \leq \mu\lambda_{max}(\mathcal{L}(G_s))$, we have: $\mu\lambda_i < \frac{\lambda_{min}(P)}{\|N\|_2}$ for all $i \in \mathcal{M}_1$. Further, since μ is a positive real number, and λ_i are non-negative real numbers, it follows that for all $i \in \mathcal{M}_1$, $\mu\lambda_i \in \mathcal{R} = \left[0, \frac{\lambda_{min}(P)}{\|N\|_2}\right)$. We have already shown that for all $\theta \in \mathcal{R}$, the matrix inequality (23) holds. Hence, by setting the uncertain parameter $\theta = \mu\lambda_i$, for all $i \in \mathcal{M}_1$, it is clear from (23) that the following set of matrix inequalities hold:

$$A_{d_{11}}X + XA_{d_{11}}^T + (1+\mu\lambda_i)N + 2\alpha_2 X \prec 0. \tag{28}$$

Hence, the set inclusion relation (20) holds (since (22) holds for all $\theta \in \mathcal{R}$), for the choice of β_i and μ as in (19), and the gain matrix $\bar{K}_1$, which satisfies the relation (18). ■

Note that according to Theorem 2, the eigenvalues of matrix: $A_{d_{11}} + \bar{\lambda}_i B_{d_1}\bar{K}_1$ belong to Ω_f if $\bar{K}_1$ satisfies matrix equation (18) and β_i, μ are chosen as specified in (19). By introducing a new variable $Y := \bar{K}_1 X$, we formulate the following LMC feasibility problem to compute X and Y.

Problem 3: For a given α_2, find $X \succ 0$, $P \succ 0$ and Y such that: $A_{d_{11}}X + B_{d_1}Y + XA_{d_{11}}^T + Y^T B_{d_1}^T + 2\alpha_2 X + P = 0$.

Once a feasible solution to Problem 3 is obtained, the matrix $\bar{K}_1$ can be computed as $\bar{K}_1 = YX^{-1}$, and thus, the feedback gain matrix to address Problem 1 is: $\bar{K} = [\bar{K}_1 \quad 0]W^T$. The computed $\bar{K}$ ensures that the finite poles of a set of descriptor subsystems (15) belong to Ω_f, and hence, the follower state trajectories will converge to the leader state trajectory with a minimum decay rate α_2 without showing impulsive behaviour. Since we have assumed that the agent dynamics are C-controllable, it follows from [20, Theorem 4] that $(A_{d_{11}}, B_{d_1})$ is completely controllable. Hence, Problem 3 has always a feasible solution. It is worth noting here that with the proposed synthesis procedure we have transformed synchronization problem of a DMAS into an eigenvalues assignment problem of a single ordinary state-space system.

C. Constant Reference Tracking

For Problem 2, each agent is represented as follows:

$$E\dot{x}_0(t) = Ax_0(t) + Bu_0(t) + w_0(t), \tag{29a}$$
$$E\dot{x}_i(t) = Ax_i(t) + Bu_i(t) + w_i(t) \text{ for } i \in \mathcal{M}_1, \tag{29b}$$

where $w_0(t)$ and $w_i(t)$ are external input signals. We choose: $w_0(t) = M\bar{r}(t)$ and $w_i(t) = g_i M_i \bar{r}(t)$, where $M \in \mathbb{R}^{n\times n}$ and $M_i \in \mathbb{R}^{n\times n}$ are feedforward gain matrices, $\bar{r}(t) = \bar{\gamma} := [\gamma_1 \ \gamma_2 \ \cdots \ \gamma_n]^T \in \mathbb{R}^n$ for $t \geq 0$ is a reference signal, and g_i is the weight associated with transferring the reference signal $\bar{r}(t)$ from leader agent to the i^{th} follower agent through neighboring agents, similar to $x_0(t)$, as shown in Figure 1(a).

Theorem 3: For the agent dynamics in (29), assume that a control architecture according to the control laws (2), (3a) and (3b), is implemented in a DMAS. Further, assume that the matrix $A_K := A + BK$ is invertible. Then, for:

$$M = -A_K, \quad M_i = -(1/g_i)A_K, \tag{30}$$

following conditions hold: $\lim_{t\to\infty}(x_0(t) - \bar{r}(t)) = 0$ and $\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0$ for all $i \in \mathcal{M}_1$.

Proof: With the control laws (2), (3a) and (3b), the agents dynamics in (29) become:

$$E\dot{x}_0(t) = A_K x_0(t) + M\bar{r}(t), \tag{31a}$$
$$E\dot{x}_i(t) = A_K x_i(t) + B\bar{K}\bar{z}_i(t) + g_i M_i \bar{r}(t). \tag{31b}$$

By taking the Laplace transformation on both sides of (31a), and using the *final value theorem* [24], it can easily be shown that for $M = -A_K$, we have: $\lim_{t\to\infty}(x_0(t) - \bar{r}(t)) = 0$. Now using the gain matrices M and M_i, as defined in (30), the dynamics in (31) can be represented as:

$$E\dot{x}_0(t) = A_K x_0(t) - A_K\bar{r}(t), \tag{32a}$$
$$E\dot{x}_i(t) = A_K x_i(t) + B\bar{K}\bar{z}_i(t) - A_K\bar{r}(t). \tag{32b}$$

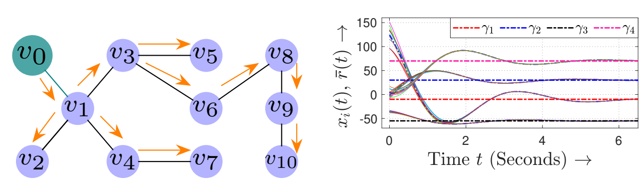
By defining: $\bar{\mathbf{r}} = [\bar{r} \ \bar{r} \ \cdots \ \bar{r}]^T$, and using the definition of $\bar{\mathbf{z}}$ as in (8), the dynamics in (32) can be written as:

$$(I_r \otimes E)\dot{\mathbf{x}}_0 = (I_r \otimes A_K)\mathbf{x}_0 - (I_r \otimes A_K)\bar{\mathbf{r}}, \tag{33a}$$
$$(I_r \otimes E)\dot{\mathbf{x}} = (I_r \otimes A_K)\mathbf{x} + (\mathcal{L}_g \otimes B\bar{K})(\mathbf{x} - \mathbf{x}_0) - (I_r \otimes A_K)\bar{\mathbf{r}}. \tag{33b}$$

Then, similar to the procedure for constructing error dynamics (12), we obtain the following error dynamics from (33): $(I_r \otimes E)\dot{\delta} = [(I_r \otimes A_K) + (\mathcal{L}_g \otimes B\bar{K})]\delta$, which is same as (12). Since $\bar{K}$, μ and β_i are designed to achieve synchronization, we have: $\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0$, for $i \in \mathcal{M}_1$, and $\lim_{t\to\infty}(x_0(t) - \bar{r}(t)) = 0$. Note that $x_i(t)$ converges to $x_0(t)$ satisfying a minimum decay rate α_2, and $x_0(t)$ converges to $\bar{r}(t)$ satisfying a minimum decay rate α_1. ■

III. Demonstrative Example

Example 1: In this example, we consider a DMAS, consisting of 11 agents, whose network graph is in Figure 2 (a). The dynamics of each agent is of the form (1), where: $E = \begin{bmatrix} 0&1&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$, $A = \begin{bmatrix} 1&2&0&0 \\ 1&3&0&1 \\ 0&-1&1&3 \\ 0&1&5&1 \end{bmatrix}$, $B = \begin{bmatrix} 0&0 \\ 1&0 \\ 0&1 \\ 0&0 \end{bmatrix}$. It is assumed that $a_{ij} = 1$, $g_1 = 1$, $g_2 = g_3 = g_4 = 0.8$, $g_5 = g_6 = g_7 = 0.6$, $g_8 = 0.4$, $g_9 = 0.2$, and $g_{10} = 0.1$. To make individual agents impulse-free and place their finite poles in Ω_l, where $\alpha_1 = 0.5$, we used the procedure in [20], and obtained the following gain matrix: $K = \begin{bmatrix} 0 & 2.2568 & 0 & -1.1318 \\ 2 & 5.3434 & 0 & -0.7478 \end{bmatrix}$. With this, the finite poles of (E, A_K) pair are: $\{-0.6148 \pm 1.7243i\}$, which lie in the chosen stability region Ω_l.



(a) (b)

Fig. 2. (a) Graph corresponding to the considered DMAS. The arrows represent the direction of flow of leader state $\boldsymbol{x_0}$ and reference signal $\bar{\boldsymbol{r}}$ to the follower agents, (b) States synchronization and reference tracking.

Now, we design the gain matrix $\bar{K}$, and the scalar gains μ and β_i such that the finite poles of the descriptor systems in (15) belong to the stability region Ω_f. We Consider the following two cases: i) *Case-1*: Ω_f is chosen with $\alpha_2 = 1$, and ii) *Case-2*: Ω_f is chosen, where $\alpha_2 = 4$. Then, Problem 3 is solved with a linear matrix inequality (LMI) solver *SeDuMi* [25], and we obtained: $\bar{K} = \begin{bmatrix} 0 & -0.0988 & 0 & -0.4654 \\ 0 & 2.7291 & 0 & 0.0276 \end{bmatrix}$, $\bar{K} = \begin{bmatrix} 0 & 4.1668 & 0 & -3.9604 \\ 0 & 10.0551 & 0 & -0.1811 \end{bmatrix}$ for Case-1 and Case-2, respectively. Further, according to Theorem 2, μ is computed as 0.0610 and 0.0319 for Case-1 and Case-2, respectively, and $\beta_i = 1/g_i,\ \forall i \in \mathcal{M}_1$. With the computed K, $\bar{K}$, β_i and μ, the closed loop DMAS (10) is simulated in MATLAB (Version R2022a) using *ode15s* solver for a set of consistent initial conditions. The states of the leader and follower agents are depicted in Figure 3 (Case-1) and Figure 4 (Case-2). It can be noticed that $x_{i_j}(t)$ converge to $x_{0_j}(t)$ within four seconds ($t_s = \frac{4}{1}$) for Case-1 and one second ($t_s = \frac{4}{4}$) for Case-2, respectively (one may refer to the *settling time*: $t_s = \frac{4}{\alpha_2}$ definition in [24]). For constant reference tracking, we consider $\alpha_1 = 0.5$ (for Ω_l) and $\alpha_2 = 1$ (for Ω_f), and $\bar{r}(t) = \begin{bmatrix} -10 & 30 & -55 & 70 \end{bmatrix}^T$. The simulated state responses are shown in Figure 2-(b). It can be observed that $x_{i_j}(t)$ are synchronized with $x_{0_j}(t)$, and then, all the states together are tracking $\bar{r}(t)$.

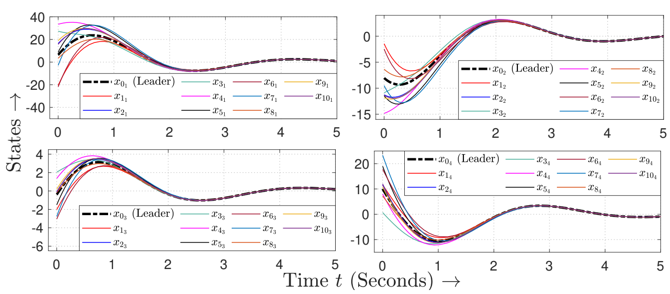


Fig. 3. Synchronization of follower states $\boldsymbol{x_{i_j}}$ (solid lines) with leader state $\boldsymbol{x_{0_j}}$ (dashed line) for Case-1.

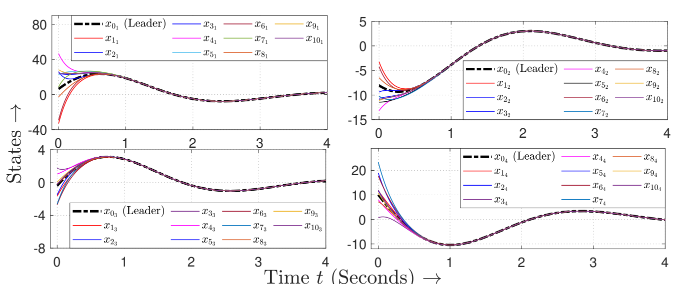


Fig. 4. Synchronization of follower states $\boldsymbol{x_{i_j}}$ (solid lines) with leader state $\boldsymbol{x_{0_j}}$ (dotted lines) for Case-2.

IV. Conclusion

We have proposed a novel control architecture to address state synchronization and reference tracking problems in a DMAS. The design procedure for selecting the underlying gain parameters is simple, where one needs to solve an LMC feasibility problem, which can efficiently be solved using existing LMI solvers, such as SeDuMi. It would be interesting to extend the proposed approach for heterogeneous agents, which however, is left for future research.

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