

Stabilization and Impulse Elimination in a Network of Impulse-Uncontrollable Descriptor Systems

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Abstract—This article explores the design of a feedback control scheme to eliminate impulses from state trajectories and simultaneously stabilize a leader-follower networked descriptor system. Assuming that the dynamics of individual descriptor systems is only *impulse-controllable*, as opposed to *impulse-uncontrollable*, we propose a distributed proportional-derivative (P-D) feedback control for follower agents and a local P-D control for the leader agent. Furthermore, we present a method for designing the proportional and derivative feedback gain vectors, which ensures that the response of the closed-loop system is stable and impulse-free. The effectiveness of the proposed control design is demonstrated through a numerical example and a networked rolling mass-spring-damper system.

Index Terms—Networked dynamical descriptor system, multi-agent system, distributed control, P-D control, impulse elimination, stabilization.

I. INTRODUCTION

In recent years, research in the field of networked dynamical systems has become increasingly popular due to their numerous applications in sectors such as power networks [1], multi-vehicle robotic systems [2], space industries [3], etc. A networked dynamical system is a group of multiple dynamical systems, known as agents, that work in a coordinated manner to achieve some collective objectives, such as consensus, internal stabilization, leader-follower synchronization, formation, and surveillance (see [2], [4]–[6] and the references therein). The agent dynamics is usually represented using ordinary differential equations. However, when agent dynamics involves differential and algebraic equations, it can effectively be represented as a *descriptor system* (DS). Examples of such systems are power networks [1], robotic manipulators [7, Section 1.2.4], biological systems [8] and cyber-physical systems [9]. A network of such systems is referred to as a *networked dynamical descriptor system* (NDDS). The state/output response of an NDDS may exhibit impulsive behavior, characterized by extremely large transient responses [7], [10]. Such behavior may cause saturation or physical damage to system components and is therefore undesirable in practical applications.

Most of the existing literature on NDDS considers consensus as a collective control objective [11]–[13]. For instance, in

[11], [12], relative output-based static and dynamic feedback control protocols were proposed for each agent to achieve consensus. Necessary and sufficient conditions were also provided for the existence of such control architectures. Authors in [13] proposed a hybrid control architecture, which was a combination of event-triggered and impulsive controllers, to achieve consensus in an NDDS. Further, the cooperative output regulation problem was also discussed in [14]. Considering signed interaction among the agents, the problem of *bipartite consensus* [15] and bipartite containment control problem [16] have also been explored. Furthermore, within a leader-follower framework, synchronization with speed regulation [17], and under actuator attacks [18] have been addressed. Most of these results, pertaining to the problems of consensus, bipartite consensus, and output regulation, assume that the individual DSs are impulse-free. Although some recent articles on leader-follower synchronization [17], [19] have relaxed this assumption, they still consider *impulse-controllability* of individual agents in order to eliminate the impulses using control laws that rely on absolute state measurements.

However, in a network of DSs, it is often more practical to utilize relative state information, as it does not rely on a global reference frame. Furthermore, the assumption of impulse-controllability is well-known to be quite restrictive in practice (see, for example, [20, Section 5.6.1], [21]). In this paper, we supplant this restrictive assumption with the more relaxed notion of *impulse-controllabilizability*. While this relaxation broadens the scope of our approach, note that in this situation, pure state feedback is rendered insufficient to eliminate impulsive responses [7, Proposition 7.1]. This necessitates the proportional-derivative (P-D) state feedback control [7, Proposition 7.3]; we utilize such a P-D feedback crucially in this paper, albeit in a distributed fashion.

The impulse elimination problem for a single DS under the assumption of impulse-controllabilizability has been well-studied in [7, Section 7.5], [22], where P-D feedback control is designed to eliminate impulses. Recently, derivative feedback was also used in the NDDS setting to increase the dynamical order of the synchronization error dynamics [23]. However, to the best of our knowledge, the problem of impulse elimination and stabilization for NDDS, when individual agents are *not* impulse-controllable, but merely impulse-controllabilizable, remains an open problem. The intricacy of the problem is further underscored by the use of relative state information.

These observations motivate us to address the impulse elimination problem in NDDS without assuming impulse-controllability of individual agents. We consider a network

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of homogeneous DSs with one leader and multiple followers, where communication is bidirectional and at least one follower is pinned to the leader. Using Weierstrass' canonical decomposition [7, Section 3.2], we decouple the NDDS dynamics into slow and fast subsystems. Since the slow subsystem is always impulse-free, we focus on the fast subsystem. To achieve an impulse-free and stable response in closed-loop NDDS, we propose a distributed P-D state feedback control for the followers and a local P-D control for the leader.

It is important to note, however, that achieving both impulse elimination and stabilization, using relative state information, introduces significant challenges. In particular, the distributed control strategy designed for impulse elimination results in closed-loop dynamics governed by the graph Laplacian, whose nonidentical eigenvalues introduce heterogeneous system modes. Hence, stabilizing the overall NDDS while using identical feedback gain for individual DSs becomes a nontrivial problem. To address this, we propose a method to design the derivative and proportional feedback gains K_d and K_p , respectively, and prove that a suitably chosen K_d ensures an impulse-free response, while K_p ensures stability.

Notation: The following notation is adopted in this article: $\otimes$: Kronecker product, $\det(A)$: determinant of matrix A , $[A]_{ij}$: $(i, j)^{th}$ element of a matrix A , $\deg(p(s))$: degree of polynomial $p(s)$, $\text{diag}\{\cdot\}$: a diagonal matrix, $\text{blkdiag}\{\cdot\}$: a block diagonal matrix, $\sigma(A, B)$: finite eigenvalues of regular matrix pair (A, B) , I_r : the identity matrix of size $(r \times r)$, $\mathbb{C}^-$: the open left half of the complex plane, $\mathbb{C}$: complex plane.

II. MATHEMATICAL PRELIMINARIES AND PROBLEM FORMULATION

A. Descriptor systems

Consider a descriptor system of the following form:

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad (1)$$

with initial condition $x(0) = x_0 \in \mathbb{R}^n$, where $E \in \mathbb{R}^{n \times n}$ is a singular matrix, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}$ are the state and input vectors, respectively. The DS (1) (or the matrix pair (E, A)) is called *regular* if the polynomial, $p(s) := \det(sE - A)$ is not identically equal to zero, that is, $p(s) \not\equiv 0$. The set of finite eigenvalues of the system (1) are the *roots* of the polynomial $p(s)$, that is, $\sigma(E, A) := \{\lambda \mid \lambda \in \mathbb{C}, p(\lambda) = 0\}$.

Definition 1: [10, Section 1-4] A linear descriptor system of the form (1) is said to be *impulse-free* if its zero-input state response, starting from any initial condition, does not contain Dirac-impulse terms.

Proposition 1: [7, Theorem 7.1 & 7.2] For the regular descriptor system (1), the following statements are equivalent:

- 1) The system is impulse-free.
- 2) $\text{rank} \begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = n + \text{rank}(E)$.
- 3) $\deg\{\det(sE - A)\} = \text{rank}(E)$.

Definition 2: [10, Definition 2-2.3] The regular DS (1) is *impulse-controllable* if, for any vector $x_0 \in \mathbb{R}^n$, there exists a sufficiently smooth control input $u(t)$ such that the terms containing the impulsive part in the response of (1) vanish for all $t \geq 0$.

Proposition 2: [10, Theorem 2-2.3] The regular DS (1) is *impulse-controllable* if and only if the following relation holds: $\text{rank} \begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = n + \text{rank}(E)$

Definition 3: [7, Definition 7.2] The regular DS (1) is said to be *impulse-controllable* if there exists a control gain vector K_d , such that the system (1), with control input $u(t) = K_d \dot{x}(t)$, is impulse-controllable.

Proposition 3: [10, Theorem 3-2.4], [7, Theorem 7.10] The regular DS (1) is *impulse-controllable* if and only if the following relation holds: $\text{rank} \begin{bmatrix} E & 0 & 0 & B \\ A & E & B & 0 \end{bmatrix} = n + \text{rank} \begin{bmatrix} E & B \end{bmatrix}$.

B. Graph theory

Consider an undirected connected graph $G(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \{v_0, v_1, \dots, v_r\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ are the sets of vertices and edges, respectively. Let a graph $\bar{G}(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ be constructed from the graph $G(\mathcal{V}, \mathcal{E})$ by removing the vertex v_0 and the edges, which are incident on v_0 . Clearly, $\bar{G}(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ is a subgraph of the graph $G(\mathcal{V}, \mathcal{E})$, that is, $\bar{\mathcal{V}} \subset \mathcal{V}$ and $\bar{\mathcal{E}} \subset \mathcal{E}$. The neighborhood of a vertex v_i in graph $\bar{G}(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ is given as: $\mathcal{N}_i := \{v_j \in \bar{\mathcal{V}} \mid (v_i, v_j) \in \bar{\mathcal{E}}\}$. The *weight of the edge*, $w_{ij} > 0$, between the vertices pair (v_i, v_j) and (v_j, v_i) are the same, that is, $w_{ij} = w_{ji}$, and $w_{ii} = 0$ (no self-loops). Then, an *adjacency* matrix $\mathcal{A}(\bar{G}) \in \mathbb{R}^{r \times r}$ is defined for $\bar{G}(\bar{\mathcal{V}}, \bar{\mathcal{E}})$, where its $(i, j)^{th}$ element, as: $[\mathcal{A}(\bar{G})]_{ij} = \begin{cases} w_{ij}, & \text{if } (v_i, v_j) \in \bar{\mathcal{E}}, \\ 0, & \text{otherwise.} \end{cases}$ Further,

a *degree* matrix $\mathcal{D}(\bar{G})$ of $\bar{G}(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ is defined as: $\mathcal{D}(\bar{G}) := \text{diag}\{d_1, d_2, \dots, d_r\}$, where $d_i := \sum_{j \in \mathcal{N}_i} w_{ij}$ for $j \neq i$ is weighted degree of vertex v_i . A *Laplacian* matrix $\mathcal{L}(\bar{G})$ of $\bar{G}(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ is defined as: $\mathcal{L}(\bar{G}) := \mathcal{D}(\bar{G}) - \mathcal{A}(\bar{G})$ [24].

C. Problem Formulation

Consider a network composed of $r + 1$ identical regular descriptor systems. The dynamics of the i^{th} descriptor system can be expressed as follows:

$$\bar{E}\dot{\bar{x}}_i(t) = \bar{A}\bar{x}_i(t) + \bar{B}u_i(t), \quad (2)$$

for $i = 0, 1, \dots, r$, where $\bar{E} \in \mathbb{R}^{\bar{n} \times \bar{n}}$ is a *singular matrix*, $\bar{x}_i(t) \in \mathbb{R}^{\bar{n}}$ and $u_i(t) \in \mathbb{R}$ are the state and input vectors of the i^{th} agent, respectively. For the NDDS (2), there exist two nonsingular matrices $S \in \mathbb{R}^{\bar{n} \times \bar{n}}$ and $T \in \mathbb{R}^{\bar{n} \times \bar{n}}$ [7, Section 3.2], [10, Section 1-3.1], such that the following Weierstrass canonical decomposition of NDDS (2), for $i = 0, 1, \dots, r$ can be obtained:

$$\dot{x}_{s_i}(t) = Jx_{s_i}(t) + B_s u_i(t), \quad (3)$$

$$N\dot{x}_i(t) = x_i(t) + B u_i(t), \quad (4)$$

where

$$S\bar{E}T = \begin{bmatrix} I_{n_s} & 0 \\ 0 & N \end{bmatrix}, \quad S\bar{A}T = \begin{bmatrix} J & 0 \\ 0 & I_n \end{bmatrix}, \quad S\bar{B} = \begin{bmatrix} B_s \\ B \end{bmatrix},$$

and $T^{-1}\bar{x}_i = [x_{s_i}^T \ x_i^T]^T$ with $n_s + n = \bar{n}$. The subsystem (3) is a standard linear system, which guarantees an impulse-free response for arbitrary initial conditions. However, subsystem (4) exhibits an impulsive response, as detailed in [7, Section 3.4], [10, Section 1-4]. Therefore, we focus on the subsystem (4) to address our problem.

Considering the dynamics in (4), it is assumed that the agent indexed by $i = 0$ is the *leader*, and the remaining r agents for $i = 1, 2, \dots, r$ are *followers*. The interactions among the agents are assumed to be bidirectional and represented using a fixed undirected graph. We are interested in addressing the following problem.

Problem 1: For a given NDDS with leader and follower dynamics as in (4), design feedback control protocols $u_i(t)$ for all the agents such that the following objectives are achieved in the response of the closed-loop system.

- i) The state response $x_i(t)$ for $i = 0, 1, \dots, r$ is free from Dirac-impulse terms.
- ii) The finite eigenvalues of the closed-loop system lie in $\mathbb{C}^-$.

III. IMPULSE ELIMINATION AND STABILIZATION OF NDDS

To address Problem 1, we consider the following assumptions on the dynamics of each agent represented in (4):

- A1** The matrix $N \in \mathbb{R}^{n \times n}$ is a Jordan nilpotent matrix with index n ($N^n = 0$ and $N^{n-1} \neq 0$). Therefore, all its entries on the first superdiagonal are 1, and all other entries are 0.
- A2** The triad (N, I_n, B) is *impulse-controllabilizable*.

Note that assumption **A1** is not restrictive in nature, as there exists a similarity transformation matrix [25, Lemma 8.40] which can transform a nilpotent matrix of index n to the Jordan nilpotent form used in assumption **A1**. On the other hand, assumption **A2**, as mentioned earlier, relaxes the restrictive assumption of impulse-controllability. Recall that this allows the use of an appropriate feedback mechanism to modify the system dynamics so as to ensure impulse-controllability of the closed-loop system.

Proposition 4: Consider the NDDS (4), where the matrix N satisfies assumption **A1**. Let the input matrix be $B = [b_1 \ b_2 \ \dots \ b_{n-1} \ b_n]^T$. Then the triad (N, I_n, B) is *impulse-controllable* if and only if $b_n \neq 0$.

Proof: Since N satisfies Assumption **A1**, it is a Jordan nilpotent matrix of index n with $\text{rank}(N) = n-1$. Consider the triad (N, I_n, B) . By defining $B_{2n} := [b_2 \ \dots b_{n-1}]^T \in \mathbb{R}^{n-2}$, the following relation holds:

$$\text{rank} \begin{bmatrix} N & 0 & 0 \\ I_n & N & B \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & I_{n-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & I_{n-2} & 0 & 0 & 0 & I_{n-2} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} B_{1n} \\ b_{n-1} \\ 0 \\ b_1 \\ B_{2n} \\ b_n \end{bmatrix}. \quad (5)$$

In (5), carrying out necessary elementary column operations with the pivots, as highlighted in the matrix, to create zeros in the respective rows, we obtain:

$$\text{rank} \begin{bmatrix} N & 0 & 0 \\ I_n & N & B \end{bmatrix} = (n-2) + 1 + 1 + (n-2) + 1 = 2n-1,$$

if and only if $b_n \neq 0$. Since $\text{rank}(N) = n-1$, it follows that $\text{rank} \begin{bmatrix} N & 0 & 0 \\ I_n & N & B \end{bmatrix} = n + \text{rank}(N)$. On the other hand, if $b_n = 0$, then $\text{rank} \begin{bmatrix} N & 0 & 0 \\ I_n & N & B \end{bmatrix} = 2n-2$, which is less than $n + \text{rank}(N)$. Hence, according to Proposition 2, (N, I_n, B) is impulse-controllable if and only if $b_n \neq 0$. ■

We are interested in the case where impulse-controllability does not hold, which is equivalent to $b_n = 0$, as shown in Proposition 4 above. In that scenario, the following result shows that $b_{n-1} \neq 0$ is a necessary and sufficient condition for *impulse-controllabilizability*.

Proposition 5: Consider the system (4). Let the input matrix be of the form $B = [b_1 \ \dots \ b_{n-1} \ 0]^T$. Then, the triad (N, I_n, B) is impulse-controllabilizable if and only if $b_{n-1} \neq 0$.

Proof: Define $B_{1n} := [b_1 \ \dots \ b_{n-2}]^T \in \mathbb{R}^{n-2}$. Under assumption **A1**, the triad (N, I_n, B) satisfies:

$$\text{rank} \begin{bmatrix} N & 0 & 0 & B \\ I_n & N & B & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & I_{n-2} & 0 & 0 & 0 & 0 & B_{1n} \\ 0 & 0 & 1 & 0 & 0 & 0 & b_{n-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & b_1 \\ 0 & I_{n-2} & 0 & 0 & 0 & I_{n-2} & B_{2n} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (6)$$

In (6), using the similar elementary column operations as in (5), we have: $\text{rank} \begin{bmatrix} N & 0 & 0 & B \\ I_n & N & B & 0 \end{bmatrix} = (n-2) + 1 + 1 + (n-2) + 1 = 2n-1$, if and only if $b_{n-1} \neq 0$. Furthermore, it is easy to see that $\text{rank}[N \ B] = n-1$. Thus, we have: $\text{rank} \begin{bmatrix} N & 0 & 0 & B \\ I_n & N & B & 0 \end{bmatrix} = n + \text{rank}[N \ B] = 2n-1$. Moreover, if $b_{n-1} = 0$, we have: $\text{rank} \begin{bmatrix} N & 0 & 0 & B \\ I_n & N & B & 0 \end{bmatrix} = 2n-2$, which is less than $n + \text{rank}[N \ B]$. Hence, by Proposition 3, (N, I_n, B) is impulse-controllabilizable if and only if $b_{n-1} \neq 0$. ■

We assume throughout the remaining part of the paper that $b_{n-1} \neq 0$ and $b_n = 0$, since we are interested in the case where (N, I_n, B) is not impulse-controllable but impulse-controllabilizable.

In order to achieve the dual objectives of impulse elimination and stabilization, we now split the control input, $u_i(t)$, for $i = 0, 1, \dots, r$, into two constituent signals as:

$$u_i(t) = u_i^I(t) + u_i^S(t). \quad (7)$$

These two signals for the agents are defined as follows.

$$u_0^I(t) = -K_d \dot{x}_0(t), \quad (8a)$$

$$u_i^I(t) = K_d \sum_{j \in \mathcal{N}_i} [w_{ij} (\dot{x}_j - \dot{x}_i) + m_i (\dot{x}_0 - \dot{x}_i)], \quad (8b)$$

$$u_0^S(t) = K_p x_0(t), \quad (8c)$$

$$u_i^S(t) = -K_p \sum_{j \in \mathcal{N}_i} [w_{ij} (x_j - x_i) + m_i (x_0 - x_i)]. \quad (8d)$$

Recall that $i = 0$ corresponds to the leader agent, while $i = 1, 2, \dots, r$ correspond to the follower agents. In (8), $K_d \in \mathbb{R}^{1 \times n}$ and $K_p \in \mathbb{R}^{1 \times n}$ are the derivative and proportional feedback gain vectors, respectively, w_{ij} is positive edge weight between vertices v_j and v_i , and m_i is the weight between the vertex v_0 and its i^{th} neighbor vertex v_i . Using this control input, as defined in (8a) and (8b) above, in (4) now, we get the resulting dynamics of the leader and follower agents as follows:

$$(N + BK_d) \dot{x}_0(t) = x_0(t) + Bu_0^S(t), \quad (9a)$$

$$N \dot{x}_i(t) = x_i(t) + BK_d \sum_{j \in \mathcal{N}_i} [w_{ij} (\dot{x}_j - \dot{x}_i) + m_i (\dot{x}_0 - \dot{x}_i)] + Bu_i^S(t), \quad \text{for } i = 1, 2, \dots, r. \quad (9b)$$

Define the following vectors: $\mathbf{x}_0(t) := [x_0^T \ x_0^T \ \cdots \ x_0^T]^T \in \mathbb{R}^{rn}$, $\mathbf{u}_0^s(t) := [u_0^s \ u_0^s \ \cdots \ u_0^s]^T \in \mathbb{R}^r$, $\mathbf{x}_f(t) := [x_1^T \ x_2^T \ \cdots \ x_r^T]^T$, $\mathbf{u}_f^s(t) := [u_1^s \ u_2^s \ \cdots \ u_r^s]^T$. Further, define a matrix $\mathcal{L}_g$ as: $\mathcal{L}_g := \mathcal{L} + \mathcal{M}$, where $\mathcal{L}$ is the Laplacian matrix of the subgraph $\bar{G}(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ and $\mathcal{M} = \text{diag}\{m_1, m_2, \dots, m_r\}$. The weights m_i , for $i = 1, 2, \dots, r$, are defined as follows:

$$m_i = \begin{cases} 1 & \text{if the leader agent is communicating with the} \\ & i^{\text{th}} \text{ follower agent} \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

Then, the dynamics in (9) may be represented as follows:

$$\{I_r \otimes (N + BK_d)\} \dot{\mathbf{x}}_0(t) = \mathbf{x}_0(t) + (I_r \otimes B) \mathbf{u}_0^s(t), \quad (11)$$

$$(I_r \otimes N) \dot{\mathbf{x}}_f(t) = \mathbf{x}_f(t) - (I_r \otimes BK_d)(\mathcal{L}_g \otimes I_n)(\dot{\mathbf{x}}_f - \dot{\mathbf{x}}_0) + (I_r \otimes B) \mathbf{u}_f^s(t). \quad (12)$$

Now, by grouping the terms of $\dot{\mathbf{x}}_f$ in (12), it follows that:

$$(I_r \otimes N + \mathcal{L}_g \otimes BK_d) \dot{\mathbf{x}}_f(t) - (\mathcal{L}_g \otimes BK_d) \dot{\mathbf{x}}_0(t) = \mathbf{x}_f(t) + (I_r \otimes B) \mathbf{u}_f^s(t). \quad (13)$$

Define the vectors $\mathbf{x}(t) := [\mathbf{x}_0(t)^T \ \mathbf{x}_f(t)^T]^T \in \mathbb{R}^{2rn}$ and $\mathbf{u}(t) := [\mathbf{u}_0^s(t)^T \ \mathbf{u}_f^s(t)^T]^T \in \mathbb{R}^{2r}$. Then, using (11) and (13), the NDDS with external input $\mathbf{u}(t)$ is given by:

$$\mathbf{N}_c \dot{\mathbf{x}}(t) = \mathbf{A}_c \mathbf{x}(t) + \mathbf{B}_c \mathbf{u}(t), \quad (14)$$

where

$$\mathbf{N}_c = \begin{bmatrix} I_r \otimes (N + BK_d) & 0 \\ -\mathcal{L}_g \otimes BK_d & (I_r \otimes N) + \mathcal{L}_g \otimes BK_d \end{bmatrix},$$

$$\mathbf{A}_c = \begin{bmatrix} I_r \otimes I_n & 0 \\ 0 & I_r \otimes I_n \end{bmatrix}, \quad \mathbf{B}_c = \begin{bmatrix} I_r \otimes B & 0 \\ 0 & I_r \otimes B \end{bmatrix}.$$

Note that $\mathbf{A}_c$, as defined above, is an identity matrix of size $2rn$. It follows that $\det(s\mathbf{N}_c - \mathbf{A}_c)$ is not identically zero, making the NDDS (14) regular. Indeed, otherwise $\det(s\mathbf{N}_c - \mathbf{A}_c)$ would evaluate to zero for all $s \in \mathbb{C}$. However, $\det(s\mathbf{N}_c - \mathbf{A}_c)$ evaluated at $s = 0$ is equal to $\det(-\mathbf{A}_c)$, which clearly is nonzero since $\mathbf{A}_c$ is the identity matrix, as noted above.

A. Design of K_d for Impulse Elimination

To eliminate impulsive behavior in the NDDS (14), we first present an auxiliary result in Lemma 1. Then, we design the derivative feedback gain vector K_d such that (14) is impulse-free. Moreover, in Theorem 3, we show that the external input, $\mathbf{u}(t)$, does not affect the impulse-free property of NDDS achieved through the design of K_d .

Lemma 1: The matrix $\mathcal{L}_g$ is nonsingular.

Proof: The matrix $\mathcal{L}_g$ has strongly connected property [26, Theorem 6.2.14]. Moreover, $\mathcal{L}_g$ is diagonally dominant, that is, $|\mathcal{L}_g[jj]| \geq \sum_{j \neq k} |\mathcal{L}_g[jk]|$, $\forall k = 1, 2, \dots, r$. Since $\mathcal{L}_g = \mathcal{L} + \mathcal{M}$, for the i^{th} row of $\mathcal{L}_g$, we have: $|\mathcal{L}_g[ii]| = \sum_{k=1, k \neq i}^r |\mathcal{L}_g[ik]| + m_i$. As the leader is pinned to at least one follower, there exists an i such that $m_i = 1$ (see (10)). Thus, for that i^{th} row of $\mathcal{L}_g$, we have: $|\mathcal{L}_g[ii]| > \sum_{k=1, k \neq i}^r |\mathcal{L}_g[ik]|$. By [26, Corollary 6.2.9], a strongly connected diagonally dominant matrix that has at least one strictly diagonally dominant row is nonsingular. Thus, $\mathcal{L}_g$ is nonsingular. ■

Theorem 1: Let the gain vector K_d be of the form $K_d := [k_{d1} \ k_{d2} \ \cdots \ k_{dn}]$, where $k_{di} \in \mathbb{R}$ for $i = 1, 2, \dots, n$. Then, there exists a K_d such that the NDDS (14) is impulse-free if and only if $k_{d1} \neq 0$.

Proof: It follows from [7, Theorem 7.2] that a regular DS of the form (14) is impulse-free if and only if the following relation holds:

$$\text{rank} \begin{bmatrix} \mathbf{N}_c & 0 \\ \mathbf{A}_c & \mathbf{N}_c \end{bmatrix} = 2rn + \text{rank}(\mathbf{N}_c). \quad (15)$$

Since $\mathcal{L}_g$ is a symmetric matrix, there exists an orthogonal matrix $Q \in \mathbb{R}^{r \times r}$ such that

$$Q^T \mathcal{L}_g Q := \mathcal{L}_\Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_r\}, \quad (16)$$

where λ_i for $i = 1, 2, \dots, r$ are eigenvalues of the matrix $\mathcal{L}_g$. Since $\mathcal{L}_g$ is a nonsingular matrix (see Lemma 1), we have $\lambda_i \neq 0$, $\forall i = 1, 2, \dots, r$. Define the following two representations of N and B under assumptions **A1** and **A2** (see also Propositions 4 and 5) as follows:

$$N = \begin{bmatrix} 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_{1n} \\ b_{n-1} \\ 0 \end{bmatrix}, \quad (17a)$$

$$N = \begin{bmatrix} 0 & e_1^T & 0 \\ 0 & N_{n-2} & e_{n-2} \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ B_{2n} \\ 0 \end{bmatrix}, \quad (17b)$$

where $N_{n-2} \in \mathbb{R}^{(n-2) \times (n-2)}$ is a nilpotent matrix, which satisfies assumption **A1**, $e_1 = [1 \ 0 \ \cdots \ 0]^T \in \mathbb{R}^{n-2}$, $e_{n-2} = [0 \ \cdots \ 0 \ 1]^T \in \mathbb{R}^{n-2}$, $B_{1n} = [b_1 \ b_2 \ \cdots \ b_{n-2}]^T \in \mathbb{R}^{n-2}$, $B_{2n} = [b_2 \ b_3 \ \cdots \ b_{n-1}]^T \in \mathbb{R}^{n-2}$, and $b_i \in \mathbb{R}$ for $i = 1, 2, \dots, n-1$. Note that (17a) and (17b) above show two *different* representations of the *same* matrices N and B . Furthermore, considering the orthogonal matrix Q as in (16) and the definition of $\mathbf{N}_c$ as in (14), the following relation holds:

$$\begin{aligned} \text{rank}(\mathbf{N}_c) &= \text{rank} \begin{bmatrix} Q^T \otimes I_n & 0 \\ 0 & Q^T \otimes I_n \end{bmatrix} \begin{bmatrix} I_r \otimes (N + BK_d) & 0 \\ -\mathcal{L}_g \otimes BK_d & N_{c22} \end{bmatrix} \begin{bmatrix} Q \otimes I_n & 0 \\ 0 & Q \otimes I_n \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} Q^T \otimes (N + BK_d) & 0 \\ -(Q^T \mathcal{L}_g Q) \otimes BK_d & (Q^T Q \otimes N) + (Q^T \mathcal{L}_g Q \otimes BK_d) \end{bmatrix}. \end{aligned}$$

Using (16) and the fact that $Q^T Q = I_r$, we have:

$$\begin{aligned} \text{rank}(\mathbf{N}_c) &= \text{rank} \begin{bmatrix} I_r \otimes (N + BK_d) & 0 \\ -\mathcal{L}_\Lambda \otimes BK_d & (I_r \otimes N) + (\mathcal{L}_\Lambda \otimes BK_d) \end{bmatrix} \\ &= r \text{ rank} \begin{bmatrix} N + BK_d & 0 \\ -\lambda_i BK_d & N + \lambda_i BK_d \end{bmatrix}. \end{aligned} \quad (18)$$

Now, using the structure of matrices N and B , as in (17a), the rank relation in (18) may be expressed as follows:

$$\begin{aligned} \text{rank}(\mathbf{N}_c) &= r \text{ rank} \begin{bmatrix} B_{1n} k_{d1} & I_{n-2} & 0 & 0 & 0 & 0 \\ b_{n-1} k_{d1} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda_i B_{1n} k_{d1} & 0 & 0 & \lambda_i B_{1n} k_{d1} & I_{n-2} & 0 \\ -\lambda_i b_{n-1} k_{d1} & 0 & 0 & \lambda_i b_{n-1} k_{d1} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= r \text{ rank} \begin{bmatrix} 0 & I_{n-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ &= r(n-2+1+n-2+1) \\ &= 2rn - 2r. \end{aligned} \quad (19)$$

Define the following matrices: $\bar{Q} := \text{blkdiag}\{Q \otimes I_n, Q \otimes I_n, Q \otimes I_n, Q \otimes I_n\}$, $N_{c22} := (I_r \otimes N) + (\mathcal{L}_g \otimes BK_d)$, $\bar{N}_{c22} :=$

$(I_r \otimes N) + (\mathcal{L}_\Lambda \otimes BK_d)$. Note that $\bar{Q}$ is an orthogonal matrix. Then, the following rank relation holds:

$$\begin{aligned} \text{rank} \begin{bmatrix} \mathbf{N}_c & 0 \\ \mathbf{A}_c & \mathbf{N}_c \end{bmatrix} &= \text{rank} \bar{Q}^T \begin{bmatrix} \mathbf{N}_c & 0 \\ \mathbf{A}_c & \mathbf{N}_c \end{bmatrix} \bar{Q} \\ &= \text{rank} \bar{Q}^T \begin{bmatrix} I_r \otimes (N+BK_d) & 0 & 0 & 0 \\ -\mathcal{L}_g \otimes BK_d & N_{c22} & 0 & 0 \\ I_r \otimes I_n & 0 & I_r \otimes (N+BK_d) & 0 \\ 0 & I_r \otimes I_n & -\mathcal{L}_g \otimes BK_d & N_{c22} \end{bmatrix} \bar{Q} \\ &= \text{rank} \begin{bmatrix} I_r \otimes (N+BK_d) & 0 & 0 & 0 \\ -\mathcal{L}_\Lambda \otimes BK_d & N_{c22} & 0 & 0 \\ I_r \otimes I_n & 0 & I_r \otimes (N+BK_d) & 0 \\ 0 & I_r \otimes I_n & -\mathcal{L}_\Lambda \otimes BK_d & N_{c22} \end{bmatrix}. \end{aligned} \quad (20)$$

Upon expanding the Kronecker products in (20), the diagonal block matrices $N+BK_d$, $N+\lambda_i BK_d$ appear r times along the diagonal. This implies (20) may be rewritten as follows:

$$\begin{aligned} \text{rank} \begin{bmatrix} \mathbf{N}_c & 0 \\ \mathbf{A}_c & \mathbf{N}_c \end{bmatrix} &= r \text{rank} \begin{bmatrix} N+BK_d & 0 & 0 & 0 \\ -\lambda_i BK_d & N+\lambda_i BK_d & 0 & 0 \\ I_n & 0 & N+BK_d & 0 \\ 0 & I_n & -\lambda_i BK_d & N+\lambda_i BK_d \end{bmatrix}. \end{aligned} \quad (21)$$

Let $k_{d_1} \neq 0$. Then choosing $K_d = [k_{d_1} \ 0 \ \cdots \ 0]$, and considering the structures of N and B as in (17a), (17b), the rank relation in (21) can be expressed as in (22) (see next page). Scanning through the rows of the matrix in (22), we get:

$$\begin{aligned} \text{rank} \begin{bmatrix} \mathbf{N}_c & 0 \\ \mathbf{A}_c & \mathbf{N}_c \end{bmatrix} &= r \left(\underbrace{n-2}_{\text{Row}_1 \text{Col}_2} + \underbrace{1}_{\text{Row}_2 \text{Col}_{1,3}} + \underbrace{n-2}_{\text{Row}_4 \text{Col}_5} \right. \\ &\quad + \underbrace{1}_{\text{Row}_5 \text{Col}_{1,4,6}} + \underbrace{1}_{\text{Row}_7 \text{Col}_8} + \underbrace{n-2}_{\text{Row}_8 \text{Col}_{8,9}} + \underbrace{1}_{\text{Row}_9 \text{Col}_3} \\ &\quad \left. + \underbrace{1}_{\text{Row}_{10} \text{Col}_{11}} + \underbrace{n-2}_{\text{Row}_{11} \text{Col}_{11,12}} + \underbrace{1}_{\text{Row}_{12} \text{Col}_6} \right) \\ &= r(4(n-2) + 6) = 4rn - 2r. \end{aligned} \quad (24)$$

In (24), $\text{Row}_i \text{Col}_j$ denotes the corresponding rank contribution from the entry in column j of the i^{th} row, whereas $\text{Row}_i \text{Col}_{j,k}$ denotes a joint rank contribution from the entries in columns j and k of the i^{th} row.

From (19) and (24), it is clear that the rank relation in (24) equals $2rn + \text{rank}(\mathbf{N}_c)$. Thus, the equivalent condition for impulse elimination, as in (15), is satisfied with the chosen K_d . This establishes sufficiency.

Next, consider $k_{d_1} = 0$, leading to $K_d = [0 \ K_{d_{2n}} \ k_{d_n}]$, where $K_{d_{2n}} = [k_{d_2} \ \cdots \ k_{d_{n-1}}] \in \mathbb{R}^{1 \times (n-2)}$. Furthermore, we define the following notations:

$$\begin{aligned} B_{1n} K_{d_{2n}} &:= B^* K^*, \quad B_{1n} k_{d_n} := B^* k^*, \quad b_{n-1} K_{d_{2n}} := b^* K^*, \\ b_{n-1} k_{d_n} &:= b^* k^*, \quad e_{11}^* := e_1^T + \lambda_i b_1 K_{d_{2n}}, \quad e_{12}^* := \lambda_i b_1 k_{d_n}, \\ e_{21}^* &:= N_{n-2} + \lambda_i B_{2n} K_{d_{2n}}, \quad e_{22}^* := e_{n-2} + \lambda_i B_{2n} k_{d_n}. \end{aligned}$$

Then, using (17a) and (17b), the rank relation in (21) may be expressed as in (23) (see next page). Scanning through the

rows of the matrix in equation (23), we get:

$$\begin{aligned} \text{rank} \begin{bmatrix} \mathbf{N}_c & 0 \\ \mathbf{A}_c & \mathbf{N}_c \end{bmatrix} &= r[(n-2) + 1 + (n-2) + 1 + 1 \\ &\quad + (n-2) + 1 + (n-2)] \\ &= 4rn - 4r. \end{aligned} \quad (25)$$

Using (19) and (25), it is clear that the rank of the matrix in (25) is less than $2rn + \text{rank}(\mathbf{N}_c)$. This implies that $k_{d_1} \neq 0$ is necessary for impulse elimination. ■

B. Design of K_p for Stabilization of NDSS

We now utilize the control signal $\mathbf{u}(t) = [\mathbf{u}_0^s(t)^T \ \mathbf{u}_f^s(t)^T]^T$ to render the system (14) stable, where $\mathbf{u}_0^s(t)$ and $\mathbf{u}_f^s(t)$ are given by (8c) and (8d) as follows:

$$\begin{aligned} \mathbf{u}_0^s(t) &= (I_r \otimes K_p) \mathbf{x}_0, \\ \mathbf{u}_f^s(t) &= (\mathcal{L}_g \otimes K_p)(\mathbf{x}_f - \mathbf{x}_0). \end{aligned} \quad (26)$$

Using the control signal, $\mathbf{u}(t)$, the overall closed-loop system is represented as:

$$\mathbf{N}_c \dot{\mathbf{x}}(t) = \bar{\mathbf{A}}_c \mathbf{x}(t), \quad (27)$$

where $\mathbf{N}_c$ is the same as in (14), and

$$\bar{\mathbf{A}}_c = \begin{bmatrix} I_r \otimes (I_n + BK_p) & 0 \\ -\mathcal{L}_g \otimes BK_p & (I_r \otimes I_n) + (\mathcal{L}_g \otimes BK_p) \end{bmatrix}.$$

Define the polynomials $p_i(s)$ for $i = 1, 2, \dots, r$, and $p(s)$ as:

$$p_i(s) := \det[s(N + \lambda_i BK_d) - (I_n + \lambda_i BK_p)], \quad (28)$$

$$p(s) := \det[s(N + BK_d) - (I_n + BK_p)]. \quad (29)$$

Then, the following result holds.

Lemma 2: The finite eigenvalues of the matrix pair $(\mathbf{N}_c, \bar{\mathbf{A}}_c)$ are the roots of the polynomials $p(s)$ with multiplicity r and the roots of the polynomials $p_i(s)$ for $i = 1, 2, \dots, r$, that is,

$$\sigma(\mathbf{N}_c, \bar{\mathbf{A}}_c) = \text{roots} \left([p(s)]^r \prod_{i=1}^r p_i(s) \right),$$

where, $\text{roots}(\cdot)$ represents the set of roots of a polynomial.

Proof: Let us consider the closed-loop system (27). The finite eigenvalues of the matrix pair $(\mathbf{N}_c, \bar{\mathbf{A}}_c)$ are the roots of $\det(s\mathbf{N}_c - \bar{\mathbf{A}}_c) = 0$. From the structure of $\mathbf{N}_c$ and $\bar{\mathbf{A}}_c$, it follows that:

$$\begin{aligned} \det(s\mathbf{N}_c - \bar{\mathbf{A}}_c) &= \det \left(s \left((I_r \otimes N) + (\mathcal{L}_g \otimes BK_d) \right) - (I_{nr} + \mathcal{L}_g \otimes BK_p) \right) \\ &= \det \left(I_r \otimes s(N + BK_d) - I_r \otimes (I_n + BK_p) \right). \end{aligned} \quad (30)$$

Now $\mathcal{L}_g$ is a symmetric matrix. Using the orthogonal matrix Q (refer to (16)) as a similarity transformation and the definition of $p_i(s)$ in (28), we obtain:

$$\begin{aligned} \det \left(s \left((I_r \otimes N) + (\mathcal{L}_g \otimes BK_d) \right) - (I_{nr} + \mathcal{L}_g \otimes BK_p) \right) \\ = \prod_{i=1}^r \det \left(s(N + \lambda_i BK_d) - (I_n + \lambda_i BK_p) \right) = \prod_{i=1}^r p_i(s). \end{aligned} \quad (31)$$

$$\text{rank} \begin{bmatrix} \mathbf{N}_c & 0 \\ \mathbf{A}_c & \mathbf{N}_c \end{bmatrix} = r \text{ rank} \begin{bmatrix} B_{1n}k_{d_1} & I_{n-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{n-1}k_{d_1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda_i B_{1n}k_{d_1} & 0 & 0 & \lambda_i B_{1n}k_{d_1} & I_{n-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda_i b_{n-1}k_{d_1} & 0 & 0 & \lambda_i b_{n-1}k_{d_1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & b_1k_{d_1} & e_1^T & 0 & 0 & 0 & 0 \\ 0 & I_{n-2} & 0 & 0 & 0 & 0 & B_{2n}k_{d_1} & N_{n-2} & e_{n-2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\lambda_i b_1k_{d_1} & 0 & 0 & \lambda_i b_1k_{d_1} & e_1^T & 0 \\ 0 & 0 & 0 & 0 & I_{n-2} & 0 & -\lambda_i B_{2n}k_{d_1} & 0 & 0 & \lambda_i B_{2n}k_{d_1} & N_{n-2} & e_{n-2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \quad (22)$$

$$\text{rank} \begin{bmatrix} \mathbf{N}_c & 0 \\ \mathbf{A}_c & \mathbf{N}_c \end{bmatrix} = r \text{ rank} \begin{bmatrix} 0 & I_{n-2} + B^*K^* & B^*k^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b^*K^* & 1 + b^*k^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_i B^*K^* & -\lambda_i B^*k^* & 0 & I_{n-2} + \lambda_i B^*K^* & \lambda_i B^*k^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_i b^*K^* & -\lambda_i b^*k^* & 0 & \lambda_i b^*K^* & 1 + \lambda_i b^*k^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_1^T + b_1K_{d_2n} & b_1k_{d_n} & 0 & 0 \\ 0 & I_{n-2} & 0 & 0 & 0 & 0 & 0 & 0 & N_{n-2} + B_{2n}K_{d_2n} & e_{n-2} + B_{2n}k_{d_n} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\lambda_i b_1K_{d_2n} & -\lambda_i b_1k_{d_n} & 0 & e_{11}^* e_{12}^* \\ 0 & 0 & 0 & 0 & I_{n-2} & 0 & 0 & 0 & -\lambda_i B_{2n}K_{d_2n} & -\lambda_i B_{2n}k_{d_n} & 0 & e_{21}^* e_{22}^* \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \quad (23)$$

Similarly, using the property $\det(I_r \otimes X) = \det(X)^r$, and the definition of $p(s)$ in (29), we obtain:

$$\det \left(I_r \otimes s(N + BK_d) - I_r \otimes (I_n + BK_p) \right) = \left[\det \left(s(N + BK_d) - (I_n + BK_p) \right) \right]^r = [p(s)]^r. \quad (32)$$

Substituting (31) and (32) into (30) yields:

$$\det(s\mathbf{N}_c - \bar{\mathbf{A}}_c) = [p(s)]^r \prod_{i=1}^r p_i(s).$$

Therefore, the finite eigenvalues of the matrix pencil $(\mathbf{N}_c, \bar{\mathbf{A}}_c)$ are given by $\sigma(\mathbf{N}_c, \bar{\mathbf{A}}_c) = \text{roots}([p(s)]^r \prod_{i=1}^r p_i(s))$. ■

The polynomial $p_i(s)$ in (28) can be further simplified (see Appendix for details), which is crucial for designing the proportional gain K_p , and is given by:

$$p_i(s) = \lambda_i(b_{n-1}k_{d_1})s^{n-1} + \lambda_i \sum_{j=1}^{n-2} (b_{n-1-j}k_{d_1} - \sum_{l=1}^j b_{n-1-j+l}k_{p_l})s^{n-1-j} - \lambda_i \sum_{l=1}^{n-1} (b_l k_{p_l}) - 1. \quad (33)$$

Similar to (33), the polynomial $p(s)$ in (29) is given by:

$$p(s) = (b_{n-1}k_{d_1})s^{n-1} + \sum_{j=1}^{n-2} (b_{n-1-j}k_{d_1} - \sum_{l=1}^j b_{n-1-j+l}k_{p_l})s^{n-1-j} - \sum_{l=1}^{n-1} (b_l k_{p_l}) - 1. \quad (34)$$

We now present Lemma 3 as an auxiliary result, followed by our main result on the design of a gain K_p such that the polynomials $p_i(s)$ for $i = 1, \dots, r$, and $p(s)$ are Hurwitz.

Lemma 3: Consider the polynomial $p_i(s)$ as defined in (33). Then, $p_i(s)$ is the characteristic polynomial of the unity negative-feedback closed-loop system with transfer function $G(s) = \frac{1}{p(s)+1}$ and gain $-\frac{1}{\lambda_i}$, as shown in Fig. 1.

Proof: Consider the polynomials $p_i(s)$ and $p(s)$ as defined in (33) and (34), respectively. Let the open-loop transfer

function be given as: $G(s) = \frac{1}{p(s)+1}$. Then, for the unity feedback configuration shown in Figure 1, the characteristic equation of the closed-loop system is $p(s) + 1 - \frac{1}{\lambda_i} = 0$, which is the same as setting $p_i(s) = 0$. ■

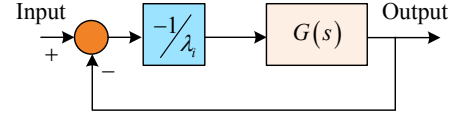


Fig. 1. Representation of $p_i(s)$ as the characteristic polynomial of a closed-loop system with gain $-\frac{1}{\lambda_i}$ and transfer function $G(s)$.

Theorem 2: Let the derivative gain, as defined in Theorem 1, be chosen as $K_d = [k_{d_1} \ 0 \ \dots \ 0]$ with $k_{d_1} = \frac{1}{b_{n-1}}$. Then, there exists K_p such that $p(s)$, $p_1(s)$, $p_2(s)$, $\dots$, $p_r(s)$ are Hurwitz.

Proof: Note that the polynomial $p(s)$ in (34) may be represented as:

$$p(s) = \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \alpha_3 s^{n-3} + \alpha_4 s^{n-4} + \dots + \alpha_{n-1} s + \alpha_n,$$

where

$$\begin{aligned} \alpha_1 &= b_{n-1}k_{d_1}, \quad \alpha_2 = b_{n-2}k_{d_1} - b_{n-1}k_{p_1}, \\ \alpha_3 &= b_{n-3}k_{d_1} - b_{n-2}k_{p_1} - b_{n-1}k_{p_2}, \\ \alpha_4 &= b_{n-4}k_{d_1} - b_{n-3}k_{p_1} - b_{n-2}k_{p_2} - b_{n-1}k_{p_3}, \\ &\vdots \\ \alpha_n &= -1 - b_1k_{p_1} - b_2k_{p_2} - b_3k_{p_3} - \dots - b_{n-1}k_{p_{n-1}}. \end{aligned} \quad (35)$$

Choosing $k_{d_1} = \frac{1}{b_{n-1}}$, the relation between α_i for $i = 2, 3, \dots, n$ and k_{p_j} for $j = 1, 2, \dots, n-1$, as given in (35), can be represented as follows:

$$\begin{bmatrix} \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \frac{b_{n-2}}{b_{n-1}} \\ \frac{b_{n-3}}{b_{n-1}} \\ \vdots \\ -1 \end{bmatrix} - \underbrace{\begin{bmatrix} b_{n-1} & 0 & 0 & \dots & 0 \\ b_{n-2} & b_{n-1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & b_3 & \dots & b_{n-1} \end{bmatrix}}_{\Theta} \begin{bmatrix} k_{p_1} \\ k_{p_2} \\ \vdots \\ k_{p_{n-1}} \end{bmatrix}. \quad (36)$$

Let $b \in \mathbb{R}^+$ be arbitrary. We now show that there exists a K_p such that

$$p(s) = (s + b)^{n-1} - 1.$$

In order to do that, we expand $(s + b)^{n-1}$ using the binomial theorem and equate its coefficients with those of the polynomial $p(s)$ in (34), given by α_i . Since the coefficient matrix Θ in (36) is lower triangular with nonzero diagonal entries, b_{n-1} , it is nonsingular. Therefore, for any vector $[\alpha_2 \ \alpha_3 \ \cdots \ \alpha_n]^T$, there exists a unique K_p satisfying (36), and thus $p(s)$ equals the desired polynomial.

Further, we show that $p(s)$ as well as $p_1(s), \dots, p_r(s)$ can each be rendered Hurwitz by choosing b sufficiently large. Let $k \in \{1, \dots, r\}$. From Lemma 3, it follows that with $p(s) = (s + b)^{n-1} - 1$, we have $p_k(s) = (s + b)^{n-1} - \frac{1}{\lambda_k}$, and for $\lambda_k = 1$, we have: $p_k(s) = p(s)$. The roots of p_k are given by:

$$\text{roots}(p_k) = \left\{ -b + \frac{1}{\sqrt[n-1]{\lambda_k}}(\alpha_j + i\beta_j) \right\},$$

such that $(\alpha_j + i\beta_j)$ is an $(n-1)^{\text{st}}$ root of unity. It then follows that $p(s)$ and $p_i(s)$ for $i = 1, 2, \dots, r$ are all Hurwitz if

$$b > \frac{1}{\sqrt[n-1]{\lambda^*}} \alpha^*, \quad (37)$$

where

$$\begin{aligned} \alpha^* &:= \max \{ \alpha_j \mid \alpha_j + i\beta_j \text{ is an } (n-1)^{\text{st}} \text{ root of unity} \}, \\ \lambda^* &:= \min \{ 1, \lambda_1, \dots, \lambda_r \}. \end{aligned}$$

Note that b can be made arbitrarily large to satisfy (37). Thus by choosing K_p accordingly, we can make $p(s)$, $p_1(s), \dots, p_r(s)$ Hurwitz. ■

We next show that the design of K_p , which is used to achieve stabilization, does not impact the impulse-free property of the closed-loop NDDS.

Theorem 3: Consider the NDDS with external input $\mathbf{u}(t)$ as in (14), where the matrix pair $(\mathbf{N}_c, \mathbf{A}_c)$ is impulse-free. Let the control input $\mathbf{u}(t)$ be implemented as in (26), resulting in the closed-loop NDDS (27) with descriptor system pair $(\mathbf{N}_c, \bar{\mathbf{A}}_c)$. Then, for any proportional gain $K_p \in \mathbb{R}^{1 \times n}$ designed according to Theorem 2, the closed-loop matrix pair $(\mathbf{N}_c, \bar{\mathbf{A}}_c)$ is impulse-free.

Proof: In Theorem 1, we have shown that the matrix pair $(\mathbf{N}_c, \mathbf{A}_c)$ is impulse-free, which using Proposition 1 implies that $\deg(\det(s\mathbf{N}_c - \mathbf{A}_c)) = \text{rank}(\mathbf{N}_c)$. Consider the stabilizing gain $K_p = [k_{p1} \ \cdots \ k_{pn-1} \ k_{pn}]$. In the proof of Theorem 2, we have shown that the parameters k_{p_j} for $j = 1, \dots, n-1$, can be uniquely determined, whereas k_{pn} can be chosen arbitrarily due to the structure of the input matrix B specified in Proposition 5. Thus, polynomials $p_i(s)$ and $p(s)$ take the forms given in (33) and (34), respectively. Moreover, by Lemma 2, the characteristic polynomial of the closed-loop NDDS is given by: $\det(s\mathbf{N}_c - \bar{\mathbf{A}}_c) = [p(s)]^r \prod_{i=1}^r p_i(s)$. Since $\deg(p_i(s)) = \deg(p(s)) = n-1$, the following relation holds: $\deg(\det(s\mathbf{N}_c - \bar{\mathbf{A}}_c)) = \deg([p(s)]^r) + \sum_{i=1}^r \deg(p_i(s)) = r(n-1) + r(n-1) = 2r(n-1)$. Recall that $\text{rank}(\mathbf{N}_c) = 2r(n-1)$ from (19), it follows that $\deg(\det(s\mathbf{N}_c - \bar{\mathbf{A}}_c)) = \text{rank}(\mathbf{N}_c)$. Thus, by Proposition 1, the closed-loop matrix pair $(\mathbf{N}_c, \bar{\mathbf{A}}_c)$ is impulse-free. ■

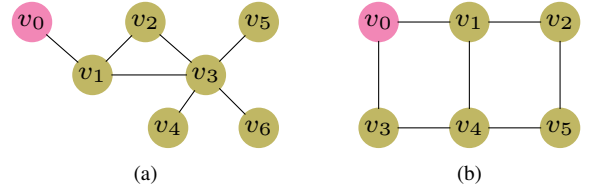


Fig. 2. Graph representations $G(\mathcal{V}, \mathcal{E})$ with (a) 7 nodes and (b) 6 nodes. In each graph, v_0 denotes the leader node, while the follower nodes are given by v_i , with $i = 1, \dots, 6$ in (a) and $i = 1, \dots, 5$ in (b).

IV. NUMERICAL EXAMPLE

Example 1: In this example, we consider a group of seven agents described by (4), consisting of one leader and six followers. The interactions among these agents are depicted by the network in Fig. 2(a). Consider edge weights $w_{ij} = 1$. It follows that the matrix $\mathcal{L}_g$, corresponding to the network in Fig. 2(a), is nonsingular with the following eigenvalues: $\lambda_1 = 0.11$, $\lambda_2 = \lambda_3 = 1$, $\lambda_4 = 1.2583$, $\lambda_5 = 3.5873$, $\lambda_6 = 6.0445$. The agent dynamics described in (4), is given by the following system matrices:

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

It is verified using [7, Theorem 7.10] that the individual DSs are impulse-controllable. To ensure an impulse-free response in the closed-loop, we choose the derivative feedback gain: $K_d = [1 \ 0 \ 0 \ 0]$ according to Theorem 1. It then follows that: $\text{rank}(\mathbf{N}_c) = 36$, and $\text{rank} \begin{bmatrix} \mathbf{N}_c & 0 \\ \mathbf{A}_c & \mathbf{N}_c \end{bmatrix} = 84$, which is equal to $2rn + \text{rank}(\mathbf{N}_c)$. Thus, NDDS (14) is impulse-free. For the considered system parameters and gains: $K_d = [1 \ 0 \ 0 \ 0]$ and $K_p = [k_{p1} \ k_{p2} \ k_{p3} \ k_{p4}]$, the polynomial $p_i(s)$ for $i = 1, \dots, 6$ is given by:

$$p_i(s) = s^3 - s^2(k_{p1}) + s(1 - k_{p2}) - (k_{p1} + k_{p3}) - \frac{1}{\lambda_i}.$$

Similarly, we obtain:

$$p(s) = s^3 - s^2(k_{p1}) + s(1 - k_{p2}) - (k_{p1} + k_{p3}) - 1,$$

and $G(s) = \frac{1}{p(s)+1}$. To place the poles of $G(s)$ in $\mathbb{C}^-$, let $p(s)+1 = (s+b)^3$, where $b > \frac{1}{\sqrt[3]{\lambda^*}} \alpha^*$ according to Theorem 2. Note that for $n = 3$, the cubic roots of unity are $\{1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}\}$. Hence, we have $\alpha^* = 1$. Furthermore, we have $\lambda^* = \lambda_1 = 0.11$ corresponding to the network in Figure 2(a). With the computed α^* and λ^* , we have $b > 2.0871$. Hence, we choose $b = 3$, and compute the gain K_p as:

$$K_p = [-9 \ -26 \ -18 \ 0],$$

using (36), which makes the closed-loop NDDS in (27) stable. Furthermore, to verify the stability and impulse-free response, the closed-loop NDDS (27) is simulated in MATLAB (Version R2024a). The state responses of the leader and follower agents are depicted in Fig. 3, where it may be observed that the states of all the agents in the network are stable and impulse-free.

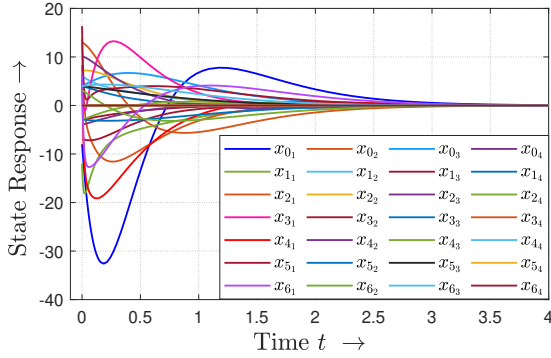


Fig. 3. Response of leader and follower states $x_i(t)$ for $i = 0, 1, \dots, 6$. x_{ij} represents the j^{th} component of each $x_i(t)$.

Example 2: Consider the dynamic model of a traction system comprising six identical wheels, each attached to a linear spring-damper unit (one of such configurations is shown in Fig. 4). We refer to this configuration as a subsystem. Let us assume that the subsystems are interconnected as shown in Fig. 2(b). Each wheel has a radius R , mass m , and rolls without slipping. The spring constant and damping coefficient are k and d , respectively. The wheel has rotational inertia J , linear displacement along the surface x , and angular velocity z . The control input u is applied as a torque at the center of the disc. The dynamics of the system shown in Fig. 4 can be completely defined by following set of equations [20, Section 5.6.1], [21]: $m\ddot{x} + d\dot{x} + kx + f_c = 0$, $J\dot{z} + Rf_c = u$, $\dot{x} = Rz$, where f_c is the contact force between the disk and the surface. Then, considering the state variable as $x = x_1$, $\dot{x} = x_2$, $z = x_3$, and $f_c = x_4$, we obtain the following set of differential and algebraic equations:

$$\begin{aligned} \dot{x}_1 &= x_2, & m\dot{x}_2 &= -kx_1 - dx_2 + x_4, \\ J\dot{x}_3 &= -Rx_4 + u, & x_2 - Rx_3 &= 0. \end{aligned} \quad (38)$$

Note that in the dynamics (38), there is an algebraic equation,

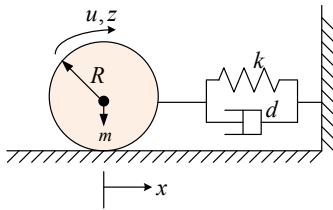


Fig. 4. A rolling disc affected by a linear spring and a damper.

$x_2 - Rx_3 = 0$, that arises from the rolling constraint (i.e., rolling without slipping) between translational and angular velocities. Thus, the above dynamics (38) is governed by a system of differential algebraic equations (DAEs) as opposed to a system of ordinary differential equations. It is well known that a system of DAEs can be represented by a descriptor system. Indeed, (38) can be represented in the form of DS (2) with the coefficient matrices as:

$$\bar{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k & -b & 0 & 1 \\ 0 & 0 & 0 & -R \\ 0 & 1 & -R & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

It is noted that for any choice of m , J , k , d , and R , the triple $(\bar{E}, \bar{A}, \bar{B})$ is not impulse-controllable, since $\text{rank} \begin{bmatrix} \bar{E} & 0 & 0 \\ \bar{A} & \bar{E} & \bar{B} \end{bmatrix} = 6$, which is not equal to $n + \text{rank}(\bar{E})$, that is, 7 [7, Theorem 4.14]. Therefore, the techniques presented in [17] cannot be applied to solve this problem.

We consider the parameters as: $k = 1$, $d = 2$, $m = 1$, $R = 2$, $J = 4$. For the system presented in (38) with the specified coefficient matrices, we find two nonsingular matrices S and T [10, Section 1-3.1] as follows:

$$S = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 1.5 & -0.5 & -1 & 0 \\ 0.75 & -0.25 & 0.5 & 0 \\ 1 & 0.5 & -5 & -1 \end{bmatrix}, \quad T = \begin{bmatrix} 0.2 & 0.4 & 0.2 & 0.3 \\ 0.6 & 0.2 & 0.1 & -0.1 \\ 0 & 0 & 0 & -0.5 \\ 0 & -0.5 & 0.25 & 2 \end{bmatrix}.$$

Using the above nonsingular matrices, we decompose the system (38) into the slow and fast subsystems as given in (3) and (4), respectively. The matrices corresponding to the slow and fast subsystems are as follows:

$$J = \begin{bmatrix} -0.5 & -0.5 \\ 0.5 & -0.5 \end{bmatrix}, \quad B_s = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.25 \end{bmatrix}.$$

It is observed that the matrices associated with the fast subsystem do not satisfy Assumption A1. However, there exists a basis transformation of the form: $x = \tilde{P}\tilde{x}$, under which the fast subsystem can be transformed into a Jordan nilpotent form [25]. Applying this transformation yields a new set of matrices for the fast subsystem, given by (for consistency with the theoretical formulation, we retain the same notation N and B for the transformed matrices):

$$N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}, \quad \text{with } \tilde{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It follows from [7, Theorem 4.14] that the NDDS (4) with the above parameters is not impulse-controllable, since $\text{rank} \begin{bmatrix} N & 0 & 0 \\ I_n & N & B \end{bmatrix} = 2$, which is not equal to $n + \text{rank}(N) = 3$. However, using Proposition 3, it is verified that the triple (N, I_n, B) is impulse-controllable as: $\text{rank} \begin{bmatrix} N & 0 & 0 & B \\ I_n & N & B & 0 \end{bmatrix} = n + \text{rank} \begin{bmatrix} N & B \end{bmatrix} = 3$. We choose the derivative feedback gain $K_d = [k_{d1} \ 0]$, as specified in Theorem 1, where $k_{d1} = \frac{1}{b_{n-1}} = 4$, in accordance with Theorem 2. We then verify that with the designed feedback gain K_d , the matrix pair (N_c, A_c) , and hence the NDDS in (14), is impulse-free. Using $K_d = [4 \ 0]$, and proportional gain $K_p = \begin{bmatrix} k_{p1} & k_{p2} \end{bmatrix}$, the resulting polynomials $p_i(s)$ for $i = 1, \dots, 5$ and $p(s)$ are given by:

$$p_i(s) = s - b_1 k_{p1} - \frac{1}{\lambda_i}, \quad p(s) = s - b_1 k_{p1} - 1.$$

To assign the poles of $G(s)$ in $\mathbb{C}^-$, we define the desired polynomial as $p(s)+1 = (s+b)$. For the network shown in Fig. 2(b), we have: $\lambda^* = \lambda_1 = 0.2841$. In this case, with $n = 1$, the only root of unity is $\{1\}$. Hence, we have $\alpha^* = 1$, and compute b as: $b > 3.5199$. Thus, we choose $b = 4$, and compute the gain K_p as: $K_p = [-16 \ 0]$, which makes the closed-loop NDDS (27) stable. To verify the stability and the absence of impulsive behavior, we simulate the closed-loop NDDS (27) in MATLAB (R2024a). The state trajectories of the leader and follower agents are illustrated in Fig. 5, demonstrating stable dynamics and the absence of impulsive behaviors in all agents.

The coordinate transformation $\tilde{x} = \tilde{P}^{-1}x$ was used to ensure that the fast subsystem satisfied Assumption A2, which

facilitated the control design. However, we performed simulations on the original system using the inverse mapping $x = \tilde{P}\tilde{x}$, thereby validating the results in the original state space.

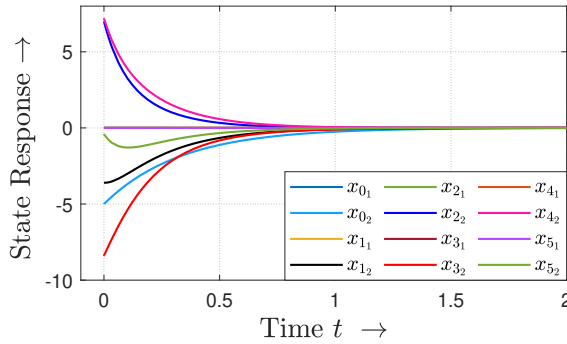


Fig. 5. Response of leader and follower states $x_i(t)$ for $i = 0, 1, \dots, 5$. x_{ij} represents the j^{th} component of each $x_i(t)$.

V. CONCLUSION

In this article, we developed a proportional-derivative feedback control for an NDDS, assuming impulse-controllability of individual agents. We presented a necessary and sufficient condition for designing the derivative control gain, which ensures that the overall NDDS is impulse-free. Furthermore, we designed a proportional feedback gain vector that guarantees stability for all the agents in the network. The proposed approach ensured that the overall closed-loop NDDS was both impulse-free and stable. The effectiveness of the proposed algorithm was demonstrated through a practical and numerical example, for which satisfactory results were obtained. Note that computing the Weierstrass form can be challenging for large-scale descriptor systems. Therefore, the proposed method is most suitable for small or medium-scale systems. It would be advantageous to translate the results presented here into a form that avoids explicit computation of the Weierstrass canonical form. This direction requires further investigations, which we plan to pursue in the near future.

APPENDIX

Construction of polynomial $p_i(s)$: Define $K_p := \begin{bmatrix} k_{p_1} & K_{p_{2n}} & 0 \end{bmatrix}$, where $k_{p_1} \in \mathbb{R}$ and $K_{p_{2n}} \in \mathbb{R}^{1 \times (n-2)} := \begin{bmatrix} k_{p_2} & \dots & k_{p_{n-1}} \end{bmatrix}$. Considering the structure of the matrices N and B as in (17b), the polynomial $p_i(s)$ in (28) is given by:

$$\begin{aligned} p_i(s) &= \det \begin{bmatrix} s\lambda_i b_1 k_{d_1} - 1 - \lambda_i b_1 k_{p_1} & s e_1^T - \lambda_i b_1 K_{p_{2n}} & 0 \\ s\lambda_i B_{2n} k_{d_1} - \lambda_i B_{2n} k_{p_1} & sN_{n-2} - I_{n-2} - \lambda_i B_{2n} K_{p_{2n}} & s e_{n-2}^T \\ 0 & 0 & -1 \end{bmatrix} \\ &= -\det \begin{bmatrix} s\lambda_i b_1 k_{d_1} - 1 - \lambda_i b_1 k_{p_1} & s e_1^T - \lambda_i b_1 K_{p_{2n}} \\ s\lambda_i B_{2n} k_{d_1} - \lambda_i B_{2n} k_{p_1} & sN_{n-2} - I_{n-2} - \lambda_i B_{2n} K_{p_{2n}} \end{bmatrix}. \end{aligned}$$

Using the property of the determinant of block matrices, the above relation may be represented as:

$$\begin{aligned} p_i(s) &= -\det(sN_{n-2} - I_{n-2} - \lambda_i B_{2n} K_{p_{2n}}) \det\{(s\lambda_i b_1 k_{d_1} \\ &\quad - 1 - \lambda_i b_1 k_{p_1}) - (s e_1^T - \lambda_i b_1 K_{p_{2n}})(sN_{n-2} - I_{n-2} \\ &\quad - \lambda_i B_{2n} K_{p_{2n}})^{-1}(s\lambda_i B_{2n} k_{d_1} - \lambda_i B_{2n} k_{p_1})\}. \end{aligned} \quad (39)$$

Using Cauchy's formula for the determinant of a *rank-one perturbation* to the matrix $sN_{n-2} - I_{n-2} - \lambda_i B_{2n} K_{p_{2n}}$, we have:

$$\begin{aligned} \det(sN_{n-2} - I_{n-2} - \lambda_i B_{2n} K_{p_{2n}}) &= \det(sN_{n-2} - I_{n-2}) (1 - K_{p_{2n}}(sN_{n-2} - I_{n-2})^{-1} \lambda_i B_{2n}) \\ &= (-1)^{n-2} (1 - \lambda_i K_{p_{2n}}(sN_{n-2} - I_{n-2})^{-1} B_{2n}). \end{aligned} \quad (40)$$

Further, using *Sherman-Morrison-Woodbury* formula, we have:

$$\begin{aligned} (sN_{n-2} - I_{n-2} - \lambda_i B_{2n} K_{p_{2n}})^{-1} &= (sN_{n-2} - I_{n-2})^{-1} \\ &\quad + \lambda_i \frac{(sN_{n-2} - I_{n-2})^{-1} B_{2n} K_{p_{2n}} (sN_{n-2} - I_{n-2})^{-1}}{1 + K_{p_{2n}}(sN_{n-2} - I_{n-2})^{-1} (-\lambda_i B_{2n})}. \end{aligned} \quad (41)$$

Define $d_i := 1 - \lambda_i K_{p_{2n}}(sN_{n-2} - I_{n-2})^{-1} B_{2n}$. Then, using (40) and (41) in (39), we get the following:

$$\begin{aligned} p_i(s) &= -(-1)^{n-2} d_i \det\{(s\lambda_i b_1 k_{d_1} - 1 - \lambda_i b_1 k_{p_1}) - \\ &\quad (s e_1^T - \lambda_i b_1 K_{p_{2n}}) \left\{ (sN_{n-2} - I_{n-2})^{-1} + \right. \\ &\quad \left. \lambda_i \frac{(sN_{n-2} - I_{n-2})^{-1} B_{2n} K_{p_{2n}} (sN_{n-2} - I_{n-2})^{-1}}{d_i} \right\} \\ &\quad \left. (s\lambda_i B_{2n} k_{d_1} - \lambda_i B_{2n} k_{p_1})\}. \end{aligned} \quad (42)$$

The relation in (42) may be further simplified as:

$$\begin{aligned} p_i(s) &= (-1)^{n-1} [(s\lambda_i b_1 k_{d_1} - 1 - \lambda_i b_1 k_{p_1}) d_i - (s e_1^T - \\ &\quad \lambda_i b_1 K_{p_{2n}}) \left\{ (sN_{n-2} - I_{n-2})^{-1} d_i + \lambda_i (sN_{n-2} - I_{n-2})^{-1} B_{2n} K_{p_{2n}} (sN_{n-2} - I_{n-2})^{-1} \right\} \\ &\quad \left. (s\lambda_i B_{2n} k_{d_1} - \lambda_i B_{2n} k_{p_1}) \right]. \end{aligned} \quad (43)$$

Now, in (43), we substitute the gain vector $K_{p_{2n}} = [k_{p_2} \dots k_{p_{n-1}}]$, and the matrices B_{1n} and B_{2n} as defined in (17a) and (17b), respectively. Then, for the system parameters, $N \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times 1}$, under assumptions **A1** and **A2**, it follows that the polynomial $p_i(s)$ in (43) may be represented as in (33).

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