The Water Jugs Problem: Solutions from Artificial Intelligence and Mathematical Viewpoints

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SOLUTIONS FROM ARTIFICIAL INTELLIGENCE AND MATHEMATICAL VIEWPOINTS

Introduction

Given a seven litre jug and a three litre jug and a water supply, can you measure out five litres of water?

I recently encountered this problem in a book on artificial intelligence. Its solution was given by setting up the problem as a production system and then searching the tree of all possible situations to locate a measurement of five litres. As a mathematician, this was not the most obvious way I would have proceeded in order to solve this problem, however this approach does exemplify a technique for problem solving much used in artificial intelligence. A comparison of the production system approach and a "mathematical" approach to the solution of this problem would thus appear to be both interesting and informative.

I hope that this article will stimulate discussion in several areas. First, using either approach to the solution, this problem could be used as a basis for a class "investigation" session in school or university. A couple of possible ideas for extending the investigation are mentioned later in the article. This particular problem has the advantage of not exhibiting some of the shortcomings of many investigations as described in Thomas (1992).

Second, the latest requirements of the National Curriculum in Mathematics explicitly state that, "Pupils should be given opportunities, where appropriate, to develop and apply their information technology (IT) capability in their study of mathematics." This problem may be solved either "mathematically" or by writing a computer program. Its alternative solutions therefore provide a platform for discussion of the similarities and differences between computer based and mathematical solutions of problems.

Third, and possibly the most interesting aspect of the comparison of the two solutions, is in the relating of the physical, real-life situation to a set of mathematical statements. Known mathematical properties of numbers are then applied to the mathematical statements to deduce a mathematical truth which may be translated back into a solution of the physical problem. Consequently, the mathematical solution in some way parallels the physical solution. This is the process we call mathematical modelling.

A major challenge to mathematics teachers at all levels is to instil an understanding of the process of mathematical modelling. The National Curriculum in Mathematics repeatedly stipulates that;

"Pupils should be given opportunities to:

- use and apply mathematics in practical tasks, in real-life problems and within mathematics itself.
- consider how algebra can be used to model real-life situations and solve problems."

Consequently, this article is offered as an example of a problem for which both a physical and a mathematical solution exist. The parallel procedures for solving the problem can thus be compared and the process of mathematical modelling may be well exemplified.

Fourth, the discussion on the comparison of the two solutions could be used as a starting point for a discussion on human problem solving and reasoning and how closely artificial intelligence techniques are able to simulate them. The production system approach, or the comparable rule-based approach, to knowledge representation and problem solving are extremely common in artificial intelligence applications. Expert systems are a widely used application of such techniques in industry and commerce. It is a challenge then to the artificial intelligence research community to go beyond the discovery of problem-solving techniques and implement problem-modelling.

The Production System Approach

The following solution is based on one for a similar problem found in Rich (1983). A more general production system solution will be described later in the article.

Let (x, y) denote the situation in which there are x litres of water in the seven litre jug and y litres of water in the three litre jug.

Initial state is (0, 0)
Goal state is (5, 0)

Production Rules

From state (x, y) there are, in general, six possible actions which may be undertaken.

1. From the water supply, fill up the seven litre jug.
2. From the water supply, fill up the three litre jug.
3. Pour the contents of the seven litre jug away.
4. Pour the contents of the three litre jug away.
5. Pour the contents of the three litre jug into the seven litre jug. (Note that there are two possible situations which could arise in this case. Either all the contents of the three litre jug will fit into the seven litre jug leaving the three litre jug empty or the seven litre jug will be filled and there will be some water remaining in the three litre jug.)
6. Pour the contents of the seven litre jug into the three litre jug. (As in the previous situation there are two possibilities to consider depending on whether the contents of the seven litre jug will fit into the three litre jug or not.)

We can express these production rules as follows.

1. If $x<7$, $\quad (x,y) \rightarrow (7,y)$
2. If $y<3$, $\quad (x,y) \rightarrow (x,3)$
3. If $x>0$, $\quad (x,y) \rightarrow (0,y)$
4. If $y>0$, $\quad (x,y) \rightarrow (x,0)$
5. If $x+y \leq 7$ and $y>0$, $\quad (x,y) \rightarrow (x+y,0)$
6. If $x+y>7$ and $x<7$, $\quad (x,y) \rightarrow (7,x+y-7)$
7. If $x+y \leq 3$ and $x>0$, $\quad (x,y) \rightarrow (0,x+y)$
8. If $x+y>3$ and $y<3$, $\quad (x,y) \rightarrow (x+y-3,3)$

Using these productions, it is possible to build up a solution space for this problem. Figure 1 shows a part of this solution space. Note that from each situation, we have used all the production rules which are applicable and applied them in ascending order. Also note that in the solution space we have not continued with any line from situations which have been developed elsewhere in the diagram or whose development is currently being explored. Such situations are underlined.

We can now use a suitable search strategy, e.g. depth-first, breadth-first, to search this solution space for (5, 0). The following solution to the problem can thus be found.

$$(0,0) \xrightarrow{1} (7,0) \xrightarrow{2} (7,3) \xrightarrow{3} (0,3) \xrightarrow{5} (3,0) \xrightarrow{2} (3,3)$$

$$\xrightarrow{5} (6,0) \xrightarrow{2} (6,3) \xrightarrow{6} (7,2) \xrightarrow{2} (7,3) \xrightarrow{3} (0,2) \xrightarrow{5}$$

$$(2,0) \xrightarrow{2} (2,3) \xrightarrow{5} (5,0)$$

This is obviously not the shortest solution since the first three steps could be replaced by one application of rule 2. However, it does establish that there is a solution and demonstrates how it can be achieved. Conversely, as the solution space is finite, (there are at most 32 possible pairs (x, y), which could possibly appear in the solution space), an exhaustive search of the solution space will determine all situations, (x, y), which can be achieved.

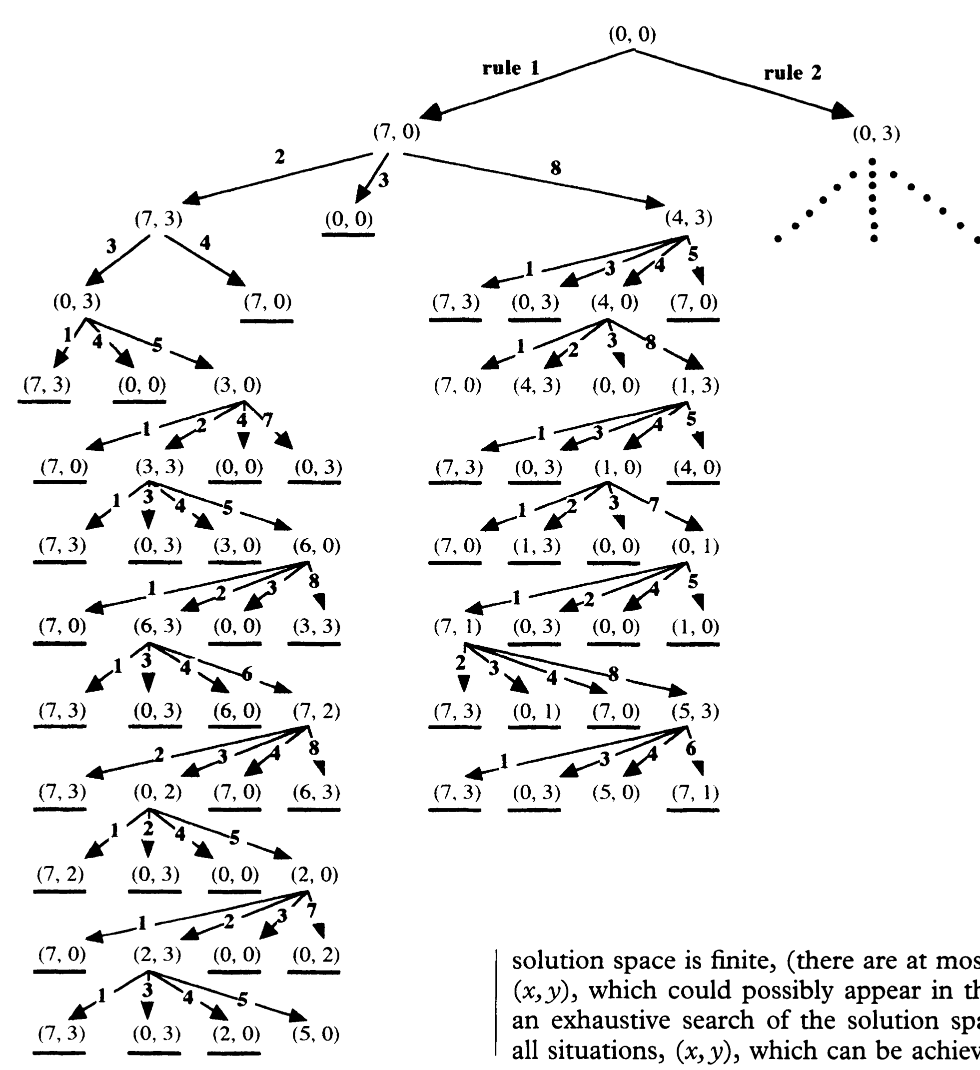

Fig. 1

The "Mathematical" Approach

The production system approach was not the one that immediately sprang to mind when I was faced with this problem. What I shall try to explain in this section is the thoughts which went through my mind when faced with this problem and the way I solved it. I cannot claim that this is the way all mathematicians would have proceeded. Neither do I claim there is anything particularly clever about my solution. It is, I feel though, quite different from the production system approach. It also exemplifies a way of working and problem solving that artificial intelligence has yet to address.

My first inclination with this problem was to try some sort of random filling and emptying of jugs. This would not be a serious attempt to solve the problem but more to get a feel for the nature of the problem. The idea of there being production rules, or something similar, would emerge but not necessarily be written down explicitly. Having seen the production system solution though I already had a good feel for the problem so I can only speculate on how much random experimenting I would have done.

However, before thinking seriously about solving the problem as stated, I reformulated the problem as follows.

Given two jugs, one which holds a litres of water and one which holds b litres of water and a water supply, is it possible to measure out c litres of water? I assume that a, b and c are positive integers.

Of course the production system approach can easily be adapted for any values of a, b and c but what I was interested in was not a technique for solving the problem for any given values for a, b and c but a formula relating a, b and c which would determine whether the problem was actually soluble. This would appear to be a significantly different way of tackling the problem yet in some ways it seems the more natural way of developing the solution.

To proceed with the solution what was required was a hypothesis. Trying out a few simple examples, the following observation was made.

If a and b are both even numbers then it is not possible to measure out an odd number of litres.

This can be generalised to the following.

If a and b are both multiples of an integer $n > 1$, then it is not possible to measure out any volume which is not a multiple of n.

The opposite of this statement can then become a reasonable hypothesis to test. If a and b are co-prime, (i.e. have 1 as their highest common factor), then it is possible to measure out any volume of water, c, where c is less than the maximum of a and b.

At this point, I used a piece of knowledge I possessed about co-prime numbers. If a and b are co-prime, there exist integers m and n such that $ma + nb = 1$. (For example, $4 \times 7 + (-9) \times 3 = 1$. Note that this is not unique, we also have, $7 + (-2) \times 3 = 1$; $(-2) \times 7 + 5 \times 3 = 1$; etc.)

Now consider the formula

$$ma + nb = 1.$$

The crucial question or hypothesis which can be constructed from this formula is something like the following.

Can I use the formula $ma + nb = 1$ to somehow show that given jugs of volumes a litres and b litres then I can measure out 1 litre of water?

In addition to solving this problem though, one also has to ask; suppose I could prove this, does it help? In other words, if I can measure out 1 litre of water, can I measure out c litres of water?

These are two distinct questions and I can give no good reason for choosing which to tackle first. They will, in fact, be considered in the order they are stated.

Suppose $ma + nb = 1$. Let's also suppose that $m > 0$. This means that n is a negative number. (Fortunately I knew that it was always possible to find m and n so that $m > 0$, but even if I had not I would have proceeded this way and considered the case $m < 0$ later if necessary.) Now suppose I have m jugs each of volume a litres and they are all full of water. I also have a number of empty jugs each of which can hold b litres of water. In fact I will require $-n$ such jugs, remember n is a negative number. I now proceed to fill the empty jugs from the m full jugs. When I have filled all the $-n$ empty b litre jugs I shall be left with just one litre of water remaining in the last of the a litre jugs.

Of course, I only actually need one b litre jug since every time it gets filled up, it is simply emptied out and used again rather than moving on to a new jug. Similarly, I only need one a litre jug since once I have emptied it I simply fill it up again from the water supply.

Hence I have a procedure for measuring out 1 litre of water.

Note that the procedure involves filling the a litre jug m times and emptying the b litre jug $-n$ times. If m is negative and n is positive then the procedure can be reversed and the a litre jug is filled from the b litre jug. The b litre jug is filled n times from the water supply and the a litre jug emptied $-m$ times. Also note that if we find the solution of $ma + nb = 1$ which gives the smallest value for $|ma|$ or $|nb|$ then we have a solution which wastes least water.

The problem is not yet finished though. I have procedure for measuring out 1 litre of water, but can I measure out c litres of water for any value of c less than the maximum of a and b?

$$\text{If} \qquad ma + nb = 1 \text{ then } (cm)a + (cn)b = c.$$

So, using the same procedure as above and assuming m is positive, c litres of water can be measured out by filling the a litre jug cm times and emptying the b litre jug $-cn$ times. If m is negative then we fill the b litre jug cn times and empty the a litre jug $-cm$ times.

Returning to the original problem, it can now be stated that as 7 and 3 are co-prime then it is possible to measure out 5 litres of water. In fact any volume of water between 1 and 6 can be measured out. We can also give a method for doing so. If the 3 litre jug is not full then pour water into it from the 7 litre jug until either it is full or the 7 litre jug is empty. When the 7 litre jug is empty, fill it up from the water supply. When the 3 litre jug is full empty it out. This procedure will eventually produce a situation in which the required amount of water is in the 7 litre jug.

Aside on Finding the Highest Common Factor

Given two integers, it is possible to find their highest common factor, and hence if they are co-prime, by the following procedure. Starting with the two given numbers, we repeatedly reduce their values by subtracting the smaller number from the larger until two numbers the same are produced. This number is the highest common factor of the two numbers we started with.

Hence, starting with 19 and 7, we have

19 7
12 7
5 7
5 2
3 2
1 2
1 1

Thus 19 and 7 are co-prime.

An interesting question, and one I leave to the reader, possibly to undertake in a class investigation, is to relate the above process to pouring water between jugs and thence to a solution of the original problem.

A Generalised Production System Approach?

O'Beirne (1965) devotes a chapter to solving variations of the water jug problem. O'Beirne would regard the production system solution as a formalised trial and error process and all his solutions are essentially production systems solutions but with a restricted set of productions. The result is thus a smaller search tree. For a variety of investigation possibilities based on this problem, including a graphical method due originally to Tweedie which first appeared in the Mathematical Gazette (1939), the reader is recommended to read O'Beirne's book.

As an example of O'Beirne's approach, the following is an alternative way of finding a solution to the original problem.

The problem can be restated in the circumstances of this approach as consisting of two jugs of capacities 3 litres and 7 litres and a third very large jug containing a large amount of water. For the purposes of this problem, this large jug will never run out of water, neither will it ever get filled up completely. Hence, pouring water from this jug is equivalent to using the water supply in the original problem and pouring water into this jug is equivalent to emptying the contents of a jug down the drain. (In fact, one can consider this third jug to be one of 10 litres capacity which starts off full but this introduces yet another degree of subtlety to the problem.)

We start with both the 3 litre and the 7 litre jug empty.

At any stage of the process, we have the three jugs designated as a source jug, a destination jug and a spare jug.

At the start, the large jug is the source jug and the other two are designated as the destination and the spare. It does not matter which is the destination and which is the spare.

The procedure is then to pour water from the source jug to the destination jug until either

the source jug is empty

or

the destination jug is full.

If a source jug becomes empty, then make the present spare jug into the new source jug and make the present source jug into the new destination jug.

If a destination jug becomes full, then make the present destination jug into the new source jug and the present spare jug into the new destination jug.

Then continue pouring from the new source jug into the new destination jug.

At first sight, this appears to be a more general production system approach to the original problem. It is certainly simpler to state than the earlier production system and it makes no mention of jug sizes. It also produces a far simpler search tree. However, there is a difficulty with this method of solution. There is no guarantee that it is going to find a solution. The set of production rules given earlier in Section 2 of this article described all possible pourings available at any time. In this formulation, one is constrained to at most two ways to proceed at any time although many more may be possible. Consequently, there is now the possibility that by just using this procedure a solution may not be found although one may exist. A not insubstantial amount of work is required to establish the validity of this procedure to produce a solution in all possible situations. This approach to solving the water jugs problem is therefore not simply a restatement of the original in more general terms but a quite sophisticated development of an algorithm for solving the problem more easily. Establishing the validity of this algorithm would make an appropriate extension of the original investigation, especially as it can be used to explain and demonstrate the meaning of mathematical proof.

(It should be noted that O'Beirne actually treats the problem in reverse. He starts from the situation one wishes to find and tries to work backwards to the original state using what he refers to as reversible pourings. Due to the reversibility of his method, there is no real virtue in proceeding in this way and achieves nothing that simply starting with two empty jugs and looking for the required solution does not. In fact, O'Beirne's method tacitly assumes the existence of a solution and is presented as a way of finding the solution rather than exploring whether a solution actually exists.)

Conclusion

It would be too presumptuous to describe my personal approach to solving this problem as "the mathematical approach". I also wonder just what reasoning processes did go through my head in formulating and solving this problem and how accurately I have been able to record them. What I do feel, however, is that the production system approach exemplifies a methodology which underlies many widely used applications of artificial intelligence whilst the mathematical approach would not find disfavour in the mathematical community.

One could suggest that the production system approach is essentially for solving a concrete problem with given values whereas the mathematical approach generalises the problem and sets more general aims. The concrete problem is, "Find a solution in these circumstances." This is exactly what the production system does. The mathematical approach is, "Is a solution possible?" The work is then at a more abstract level. The former could be described as a problem-solving approach. The latter is a problem-modelling approach. I believe there is something distinctly different in the two approaches which the subject of artificial intelligence has not yet completely identified.

The "generalised" solution lies somewhere in between the above two extremes. It is essentially a production system but requires a rigorous mathematical argument to establish its validity. However, this generalising or simplifying of the production system is itself an interesting process and the subject of complexity of algorithms has become a vast and complex area of study.

Perhaps the most interesting feature of the mathematical solution though is not its value as a solution to the problem itself but as an example of mathematical modelling. The fact that there is also a more physical solution adds to its interest. As stated in the introduction to this article, instilling an understanding of the process of mathematical modelling is one of the major challenges facing teachers of mathematics today at all levels.

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