ELL780: Minor Test I

August 29, 2017

Maximum Marks: 25

1. Let $\mathbf{V}$ be the vector space of all (2×2) real matrices, under the usual operations of matrix 'addition' and scalar 'multiplication'. Let $\mathbf{M} \subseteq \mathbf{V}$ be given by

$$\mathbf{M} = \left\{ \mathbf{A} = (a_{ij})_{2\times 2}, a_{ij} \in \mathbb{R}, \mathbf{A}^{\mathrm{T}} = \mathbf{A}, \text{ and } Trace(\mathbf{A}) = 0 \right\}.$$

- Verify that $\mathbf{M}$ is a vector subspace of $\mathbf{V}$.
- Identify a basis of $\mathbf{M}$.
- What is the dimension of $\mathbf{M}$?
- Can you state a general result with regard to the dimension of $\mathbf{M}$ if $\mathbf{V}$ happens to be the set of all $(n \times n)$ real matrices? **[6]**

2. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y + z \\ x + y - z \end{pmatrix}.$$

Let $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ be a basis for $\mathbb{R}^2$, and $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \right\}$ be a basis for $\mathbb{R}^3$.

- Find the matrix $\mathbf{A}$ of the given linear transformation T.
- Determine the $Rank(\mathbf{A}^{\mathrm{T}}\mathbf{A})$. **[6]**

3. Let $P : (x_1, y_1, z_1)$ and $Q : (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. Let

$$d_1(P, Q) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|.$$

- Show that $(\mathbb{R}^3, d_1)$ is a metric space.
- In $(\mathbb{R}^3, d_1)$, give a rough sketch of the ball of radius 1 with centre as $(0, 0, 0)$.
- Identify a Cauchy sequence in $(\mathbb{R}^3, d_1)$.
- Identify a non-convergent sequence in $(\mathbb{R}^3, d_1)$. **[6]**

4. Construct an example of each of the following (if no such example is possible, then appropriate reasons must be provided).

(i) a closed convex cone in $\mathbb{R}^3$.

(ii) a metric space (X, d) in which every set $A \subseteq X$ is open as well as closed.

(iii) a continuous mapping $T : (X, d) \longrightarrow (Y, \hat{d})$ such that $B \subseteq Y$ is open and $A = T^{-1}(B) \subseteq X$ is closed.

(iv) a set of four linearly independent vectors in $\mathbb{R}^3$.

(v) a set A in $\mathbb{R}^2$ for which $Int(A) = \emptyset$. **[7]**