ELL780: Minor Test I

August 30, 2015

Maximum Marks: 20

1. (i) Let (P) be the statement in English with (P): *function* $f : \mathbb{R} \to \mathbb{R}$ *is continuous at* x_0. Then, construct an equivalent statement (Q) using $\epsilon - \delta$ with necessary explanations, if there be any. **[2]**

   (ii) Construct the negation $(\not{Q})$ with necessary changes. Assume missing data, if there be any. **[1]**

2. Let U be the universal set and X, Y be subsets of U. Then, show that $(X \cup Y)^{\mathsf{C}} = X^{\mathsf{C}} \cap Y^{\mathsf{C}}$ and $(X \cap Y)^{\mathsf{C}} = X^{\mathsf{C}} \cup Y^{\mathsf{C}}$. **[2]**

3. (i) Consider the statement:

$$f(x_1) = f(x_2) \implies x_1 = x_2;\ \forall x_1, x_2 \in \mathcal{D}(f).$$

   Write down another statement which is equivalent to this one. **[1]**

   (ii) Let $f : \mathcal{D}(f) \subset \mathbb{R} \to \mathbb{R}$ such that $\mathcal{R}(f) \subsetneq \mathbb{R}$. Under what conditions is $f : \mathcal{D}(f) \to \mathcal{R}(f)$ invertible from $\mathcal{D}(f)$ *onto* $\mathcal{R}(f)$? Is f also invertible as a function from $\mathcal{D}(f)$ *into* $\mathbb{R}$? Justify your answer. **[1]**

4. (i) Examine the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$f(x,y) = \begin{cases} \frac{x^6}{(y-x^3)^2+x^7} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$$

   Find the limits
   (a) $\lim_{x\to 0} f(x, mx)$ for $m \in \mathbb{R}$;
   (b) $\lim_{y\to 0} f(my, y)$;
   (c) $\lim_{y\to 0} f(0, y)$;
   (b) $\lim_{x\to 0} f(x, 0)$.
   Do these results imply that $\lim_{(x,y)\to(0,0)} f(x,y)$ exists? **[3]**

   (ii) Show that $f(x) = 1 - |x|$ is continuous on $\mathbb{R}$, but not differentiable on $\mathbb{R}$. **[1]**

5. (i) Examine the function $f(x) = |1 - |x||$ on $[-1, 1]$ for relative extrema, if there be any, with justification. **[1]**

   (ii) Find the critical points of the function $f(x) = |1 - |x||$ on $[-1, 1]$. **[1]**

6. (i) State the three most important properties of the set $\mathbb{Q}$ of rational numbers. State the property which $\mathbb{Q}$ does *not* possess. **[1]**

   (ii) Let $Q_0 = \{x : x \in \mathbb{Q} \text{ s.t. } x^2 < 3\}$. Then, does there exist $\sup Q_0$ in $\mathbb{Q}$? **[1]**

7. (i) Let $f : X \to \mathbb{R}$. Define
(a) $\sup_{x\in X} f(x)$,
(b) $\inf_{x\in X} f(x)$ in $\mathbb{R}$. **[1]**

(ii) Let $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ defined by:

$$f(x) = \frac{1}{e^x}$$

Does it have a maximum in $\overline{\mathbb{R}}$? **[1]**

8. (i) Let f be a bounded real-valued function. Then, show that
(a)$\sup_{x\in X} f(x) = -\inf_{x\in X}(-f(x))$,
(b)$\inf_{x\in X} f(x) = -\sup_{x\in X}(-f(x))$. **[1]**

(ii) Suppose $f(x) \leq g(x)$, where g is bounded above; do $\sup f(x)$, $\sup g(x)$ exist and if they exist, how are they related? Prove the result. **[2]**