ELL780: Major Test

November 20, 2015

Maximum Marks: 40

1. Let $f : [0,4] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} (2-x) & , \quad 0 \le x \le 1 \\ x^2 & , \quad 1 < x \le 2 \\ (x+2) & , \quad 2 < x \le 4 \end{cases} .$$

(i) Sketch f and hence verify that f is a unimodal minimum function.

(ii) Perform two iterations of the Golden Section Rule to minimize $f(x)$ over $[0,4]$. What is the length of the search interval after 2 iterations? **[5]**

2. Let $d^{(1)} = (1,0)^T$ and $Q = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$.

(i) Find $d^{(2)} \in \mathbb{R}^2 (d^{(2)} \neq 0)$ such that $d^{(1)}$ and $d^{(2)}$ are Q-conjugate.

(ii) Use $d^{(1)}$ and $d^{(2)}$ as obtained above to solve

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 6x_2 &= 1. \end{aligned}$$

[5]

3. Consider

$$\begin{aligned} &\text{Max} \qquad z = 4x_1 + \beta x_2 \\ &\text{subject to} \\ &\qquad x_1 + x_2 \le 8 \\ &\qquad 2x_1 + x_2 \le 10 \\ &\qquad x_1 \ge 0, x_2 \ge 0. \end{aligned}$$

Use Karush-Kuhn-Tucker optimality conditions to determine all values of β for which $(\bar{x}_1 = 0, \bar{x}_2 = 8)$ is optimal. **[5]**

4. Consider the following quadratic programming problem (QPP)

$$\begin{aligned} &\text{Max} \qquad 4x_1 - 3x_2 - 3x_1^2 + 6x_1x_2 - 4x_2^2 \\ &\text{subject to} \\ &\qquad x_1 + x_2 \le 8 \\ &\qquad 2x_1 + x_2 \ge 10 \\ &\qquad x_1 \ge 0, x_2 \ge 0. \end{aligned}$$

Let the given (QPP) be solved by the Wolfe Method.

(i) Write the first tableau.

(ii) Identify the column to enter and column to leave, and hence all entries in the next tableau.

(Do NOT proceed any further even if needed.) **[5]**

5. Suppose the cumulative distribution function (c.d.f.) of a random variable X is given by

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ 1/2 & , \quad 0 \le x < 1 \\ 1 & , \quad 1 \le x < \infty \end{cases} .$$

(i) Show that X is a discrete random variable.

(ii) Obtain the probability mass function of X.

(iii) Determine $P(1/2 \le X \le 2)$. **[5]**

6. Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with probability 0.7, and if it does not rain today, then it will rain tomorrow with probability 0.4.

Evaluate the probability that it will rain three days from today given that it is raining today.

(Hint: Obtain the one-step transition probability matrix P.) **[5]**

7. In each of the following instances, is $f(x)$ uniformly continuous, or non-uniformly continuous, or discontinuous, for $x \in S$? In each case, prove your answer.

(i) $f(x) = x^2$, $S =]0, \infty[$ **[1.5]**

(ii) $f(x) = \frac{1}{x}$, $S = [0.1, \infty[$ **[1.5]**

8. Let V be the vector space consisting of all functions of the form

$$\alpha e^{2x} \cos x + \beta e^{2x} \sin x.$$

Consider the following linear transformation $L : V \to V$

$$L(f) = f' + f$$

where f' denotes the derivative of f with respect to x.

(i) Find the matrix representing L with respect to the basis $\{e^{2x} \cos x, e^{2x} \sin x\}$. **[2]**

(ii) Use your answer from part (a) to find one solution to the following differential equation:

$$y' + y = e^{2x} \cos x.$$

[2]

9. Examine the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$f(x, y) = \begin{cases} \frac{x^6}{(y - x^3) + x^7} & \text{for } (x, y) \ne (0, 0) \\ 0 & \text{for } (x, y) = (0, 0). \end{cases}$$

Find the limits

(i) $\lim_{x \to 0} f(x, mx)$ for $m \in \mathbb{R}$;

(ii) $\lim_{y \to 0} f(my, y)$;

(iii) $\lim_{y \to 0} f(0, y)$;

(iv) $\lim_{x \to 0} f(x, 0)$.

Do these results imply that $\lim_{(x,y) \to (0,0)} f(x, y)$ exists? **[3]**