Q.8. (Here $y_n \triangleq t_n$ has been used)

(a)
$$L = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi}\sigma x_n} e^{-\frac{(y_n - w x_n)^2}{2\sigma^2 x_n^2}}$$

$$\log L = \sum_n \left\{ \log(\sqrt{2\pi}\sigma) - \log x_n - \frac{(y_n - w x_n)^2}{2\sigma^2 x_n^2} \right\}$$

(b)
$$\frac{\partial \log L}{\partial w} = \sum_n \frac{\cancel{2}(y_n - w x_n)\cancel{x_n}}{\cancel{2}\sigma^2 x_n^{\cancel{2}}} = \sum_n \left(\frac{y_n}{\cancel{\sigma^2} x_n} - \frac{w}{\cancel{\sigma^2}} \right) = 0$$

(c)
$$\Rightarrow \hat{w}_{ML} = \frac{1}{N} \sum_n \frac{y_n}{x_n}$$

(d)
$$E(\hat{w}_{ML}) = \frac{1}{N} \sum_n \frac{E(y_n)}{x_n} = \frac{1}{N} \sum_n \frac{w \cancel{x_n}}{\cancel{x_n}} = \underline{w}$$

$\Rightarrow$ bias $= 0$

$$\mathrm{var}(\hat{w}_{ML}) = \mathrm{var}\left(\frac{1}{N} \sum_n \frac{y_n}{x_n} \right)$$

$$= \frac{1}{N^2} \sum_n \mathrm{var}\left(\frac{y_n}{x_n} \right) = \frac{1}{N^2} \sum_n \frac{1}{x_n^2} \mathrm{var}(y_n)$$

$$= \frac{1}{N^2} \sum_n \frac{1}{\cancel{x_n^2}} \sigma^2 \cancel{x_n^2}$$

$$= \frac{\sigma^2}{N}.$$

(e)
$$p(w \mid y, x) \propto \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi}\sigma x_n} e^{-\frac{(y_n - w x_n)^2}{2\sigma^2 x_n^2}} \cdot \frac{1}{\sqrt{2\pi\alpha}} e^{-\frac{w^2}{2\alpha}}$$

$$\log p = \sum_n \left\{ -\log(\sqrt{2\pi}\,\sigma) - \log x_n - \frac{(y_n - w x_n)^2}{2\sigma^2 x_n^2} \right\}$$

$$- \log(\sqrt{2\pi\alpha}) - \frac{w^2}{2\alpha}$$

(f) $$\frac{\partial \log p}{\partial w} = \sum_n \left(\frac{y_n}{\sigma^2 x_n} - \frac{w}{\sigma^2} \right) - \frac{w}{\alpha} = 0$$

$$\Rightarrow \cancel{\frac{1}{\sigma^2}} \sum_n \frac{y_n}{x_n} = w\left(\frac{N}{\cancel{\sigma^2}} + \frac{\sigma^2}{\alpha} \right)$$

(g) $$\hat{w}_{MAP} = \frac{1}{\left(N + \frac{\sigma^2}{\alpha}\right)} \sum_n \frac{y_n}{X_n}$$

(h) $$E(\hat{w}_{MAP}) = \frac{1}{\left(N + \frac{\sigma^2}{\alpha}\right)} \sum_n \frac{E(y_n)}{X_n} = \frac{N}{N + \frac{\sigma^2}{\alpha}} w$$

$$\Rightarrow \text{bias} = -\left(1 - \frac{N}{N + \sigma^{-2}\alpha}\right) w$$

$$\text{var}(\hat{w}_{MAP}) = \frac{1}{\left(N + \frac{\sigma^2}{\alpha}\right)^2} \sum_n \frac{1}{\cancel{X_n^2}} \cdot \sigma^2 \cancel{x_n^2}$$

$$= \frac{N}{\left(N + \frac{\sigma^2}{\alpha}\right)^2} \sigma^2$$

(i) $\alpha \uparrow$ Bias $\uparrow$ Var $\downarrow$

(j) $p\left(\frac{t}{x} \middle| x\right) = N(w, \sigma^2)$