ELL880: Minor Test I

February 7, 2023

Maximum Marks: 15

Instructions:

- **Please clearly indicate the question number, and part number if applicable, at the start of each response.**
- **Please read all questions carefully.**
- **Please ensure that your responses are to-the-point and that you write only what is asked for on the answer script you submit.**
- **While the exam is open-notes, all your answers must be written entirely in your own words, without any copying from anywhere.**
- **Please try to be clear and careful with all mathematical notation, so that there is no ambiguity in the expressions/formulae you write down. Try to stick to the kind of notation used in class as far as possible.**

1. In class we set up a statistical model for estimating the proportion of black in a black-and-white image, based on a random sample of some pixels whose colour has been observed. In that model, N (the no. of pixels sampled), N_B (the number of black pixels observed), and N_W (the number of white pixels observed) were the observed variables, while p (the proportion of black in the overall image) was the unobserved variable or parameter to be estimated.

   Now suppose that we instead have $N_B = 4$ and $p = 0.6$ as observed/known variables, but somehow N (and hence N_W) are not observed (let's say we forgot to count the total number of pixels sampled and only counted the black ones). Using the same modelling assumptions as in class, obtain the posterior distribution of N, given the observed variables. You may assume that N is at most 10. All your working should be shown clearly, and any assumptions made should be explicitly stated. **[7]**

2. Consider a simple linear regression model of the kind discussed in class:

$$W_i \sim \mathcal{N}(\mu_i, \sigma)$$
$$\mu_i = \alpha + \beta H_i$$

   When doing discriminative modelling, we can write Bayes' theorem for this model as follows, conditioning all terms on H_i:

$$P(\alpha, \beta, \sigma | W_i, H_i) = \frac{P(W_i | H_i, \alpha, \beta, \sigma) P(\alpha, \beta, \sigma | H_i)}{P(W_i | H_i)}$$

   In class we further took $P(\alpha, \beta, \sigma | H_i) = P(\alpha, \beta, \sigma)$, based on intuition. Give a more precise proof of this last equality, making use of Bayes' theorem once again. All steps in your working should be clearly shown/justified. **[4]**

3. In class we developed a model to study the effects of sex and height on weight. Now suppose we add in another observed variable, age. We would like to allow for the possibility that age could influence both height and weight.

   (a) Draw a full DAG representing the updated causal structure of the generative model after including age. **[2]**

   (b) Suppose $\boldsymbol{\theta}$ represents some set of parameters in a generative model for *height*, based on the above DAG. Write down the general expression for the posterior distribution of $\boldsymbol{\theta}$. **[2]**