Source-Aware Adaptive Power Allocation in OFDM Systems for Rate Constrained Applications

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***Abstract*—In a wireless OFDM system, due to channel fading all subcarriers are not always usable for an acceptable performance. In this paper, we propose two source-aware and channel-adaptive OFDM resource allocation approaches that judiciously selects the optimum required subcarriers. The first scheme increases the power efficiency while maintaining a chosen reception quality, whereas the second scheme shows a significant improvement in reception quality for a given power budget.**

***Index Terms*—Adaptive OFDM system, DWT source, power allocation, distortion minimization, power minimization**

I. Introduction

In OFDM systems, frequency selectivity across the subcarriers allows to increase the performance by assigning different powers and rates according to channel gain to noise ratio (CNR). Water-filling approach assigns different rates and powers to different subcarriers to achieve the same bit error rate (BER) [1]. Similarly, for goodput and fairness maximization in an OFDMA system, [2] applies greedy power allocation that supports different modulation rates over different subcarriers. These approaches require a high signaling overhead in fast fading environment. To reduce the overhead and computational complexity, subcarriers are grouped and the resources are allocated in clusters [3], [4]. The performance loss in such strategies can be severe in frequency selective channels.

A user-end quality maximization approach subject to the total transmit power constraint in multiuser OFDM systems [5] was shown to do better than water-filling algorithms. However, the power adaptation does not account the source importance. In an alternative scheme [6] (henceforth called *constant rate adaptation (CRA)*), rates allocated to the currently used subcarriers (the set N_u with cardinality n) are the same, i.e., $r_i = r, \forall\ i \in N_u$. The assigned power p_i is adjusted so that, though different subcarriers may have different BERs ε_i, the system requires a minimum total power $\left(\text{minimize } \sum_{i\in N_u} p_i\right)$ to achieve an acceptable average BER ε, i.e., $\sum_{i\in N_u} \varepsilon_i \leq n\varepsilon$.

The above source-oblivious approaches may not offer optimal reception quality. Specifically, the CRA scheme [6] that aims to achieve an average BER over the used subcarrier set may be good for sending equally important data blocks (e.g., symmetric multiple description coding (MDC)), but it does not exploit diversity for the sources (e.g., image/video) having unequal importance. This is because, the power allocation could be such that, the assigned subcarrier to a higher (respectively, lesser) important data may have a high (respectively, low) BER, thereby degrading the reception quality.

A source-aware power adaptation strategy in [7] considered single channel resource. To minimize the total distortion, a power allocation scheme was proposed in [8] that assigns power optimally at coding pass (CP) levels of a JPEG2000 bitstream. The algorithm, henceforth called *CP based unequal power allocation (CPUPA)*, categorizes all the CPs of an image in L different groups and allocates same power to all the bits in a group, but the power allocation from one group to another may vary. The granularity and efficiency of the power allocation algorithm is limited by the number of groups L.

Joint source-channel adaptation has been addressed by several researchers, which consider different source-based OFDM systems, e.g., MDC, DCT (discrete cosine transform), and DWT (discrete wavelet transform). MDC-OFDM systems [9] have their own complexities in optimal redundancy assignment and joint physical layer adaptation. In [10] source-aware subcarrier power allocation was proposed for DWT video transmission in multiuser OFDMA systems. This approach was extended to cognitive radio networks in [11], where interference to the primary users was of interest. In these studies, due to quantized feedback, optimum subcarrier power allocation is governed by (limited to) white Gaussian noise rather than the channel fading. Also, these studies did not consider channel rate adaptation and end-user rate constraint.

We consider a DWT-OFDM system to illustrate the concept of source-aware resource allocation with CNR feedback, where, unlike in conventional layered coded transmission, retransmission of lost packets are not allowed. Mapping of source content and power adaptation in a rate-constrained multichannel scenario, as in our case, makes it an entirely different system challenge with respect to the approach in [7].

Our key contributions are: (1) For the resource allocation performance study a new generalized polynomial mapping function is defined to capture the varying importance levels of the source. (2) Two source-aware constant-rate adaptive techniques (*Schemes I and II*) are proposed; Scheme I increases power efficiency appreciably compared to reference schemes while maintaining the same reception quality, whereas Scheme II improves the reception quality for the same power consumption as in the reference schemes. These schemes have a reduced signaling overhead as in CRA [6], while the optimality is achieved in a reduced search space and run time.

II. System model

A. Adaptive DWT-OFDM system

To illustrate the proposed idea of source-aware resource allocation, DWT compressed source data vectors are trans-

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mitted using adaptive OFDM system with N subcarriers in fading environment. For a fair comparison with the competitive approaches [6], [8] and also motivated by the channel estimation in [12], a perfect CNR feedback is assumed available at the transmitter to adaptively allocate the resources among subcarriers. Similar assumption has been widely used in the literature (e.g., [13], [14]) The steps involved are as follows:

(i) DWT is applied on an image frame of original size $S_1 \times S_2$ pixels, producing four sub-images: LL, HL, LH, and HH, in decreasing order of information content.
(ii) From the sub-images, four coefficient vectors x_1, x_2, x_3, and x_4 are generated, each of length $\frac{S_1 \cdot S_2}{4}$.
(iii) The coefficient vectors are uniformly quantized with an average M bits/coefficient to form four bit streams.
(iv) Each bit stream is chopped into bit vectors of size N' bits. Four such bit vectors (of unequal importance) are grouped, which we call group of vectors (GOVs).

Without loss of generality, we restrict to first-level compression. We take an example DWT-OFDM system with FFT size $N = 256$ and bit vector size $N' = 512$ bits. For this system, 64 GOVs are arranged in parallel to obtain 256 bit streams, to be simultaneously transmitted through different subcarriers.

Adaptive power allocation may designate a set of unused subcarriers. Data mapped on to those subcarriers are discarded at the transmitter, and are replaced at the receiver by the average coefficient values of the respective sub-images, introducing some distortion. To reduce this distortion, in the proposed schemes the less important data (high pass components) are mapped to the subcarriers having low CNR. The loss of data due to channel errors also contribute to the distortion.

B. Model to generate general compressed source vectors

Let $\sigma^2_{x_1}$, $\sigma^2_{x_2}$, $\sigma^2_{x_3}$, and $\sigma^2_{x_4}$ are the respective variances of the coefficient vectors x_1, x_2, x_3, and x_4. By DWT compression property, $\sigma^2_{x_1} \geq \sigma^2_{x_2} \geq \sigma^2_{x_3} \geq \sigma^2_{x_4}$. The corresponding importance levels are also in descending order. For a study on different sources, a generalized mapping function is defined,

$$\sigma^2_{x_k} = \alpha(v)(k-1)^v + \beta(v), \quad \text{for } k = 1,2,3,4, \tag{1}$$

By varying the controlling parameter v, the variances of different data vectors of any given compressed image can be realized. The coefficients $\alpha(v)$ and $\beta(v)$ are chosen to control absolute values of the variances of a compressed source. The variances are normalized with respect to the highest value $\sigma^2_{x_1}$, with $\sigma^2_{x_1} = 1$, which corresponds to $\beta(v) = 1$. $\alpha(v)$ is obtained by setting $\sigma^2_{x_k} = 0$ at $k = 5$, which gives $\alpha(v) = -0.25^v$. Thus, the normalized variances are: $\sigma^2_{x_1} = 1$; $\sigma^2_{x_2} = 1 - 0.25^v$; $\sigma^2_{x_3} = 1 - 0.5^v$; $\sigma^2_{x_4} = 1 - 0.75^v$. Note that, a higher value of v corresponds to a lesser difference among the variances $\sigma^2_{x_k}$. We will concentrate on lower values of v (standard Lena image is best matched with $v = 0.02$) because a high gradient of importance is a desirable property for this system. Let, D_i be the contribution to the total mean square distortion when the data vector assigned to the ith subcarrier is unavailable at the receiver. Then, D_i is expressed as:

$$D_i = \begin{cases} \sigma^2_{x_1}, & \text{for} \quad 1 \leq i \leq N/4 \\ \sigma^2_{x_2}, & \text{for} \quad N/4 < i \leq N/2 \\ \sigma^2_{x_3}, & \text{for} \quad N/2 < i \leq 3N/4 \\ \sigma^2_{x_4}, & \text{for} \quad 3N/4 < i \leq N. \end{cases} \tag{2}$$

III. Formulation and Analysis

We consider a transmitter-receiver pair in an OFDM system with N subcarriers. Fading is assumed to be constant over a GOV transmission duration. A perfect knowledge of CNRs (g_i) is assumed available at the transmitter. N_u with cardinality n is defined as the set of used subcarriers, each of which is assigned with a non-zero power. The target application requires a minimum rate guarantee. The desired transmission rate, lower bounded by R bits per OFDM symbol, is equally distributed among the subcarriers in set N_u. Each subcarrier having rate r bits per OFDM symbol should satisfy the rate constraint $nr \geq R$. The power allocated to the ith subcarrier is p_i. User data is transmitted over the N_u subcarriers with optimum power allocation. At the receiver, distortion is introduced because of the discarded data vectors corresponding to the unused number of subcarriers ($N - n$) and also due to transmission errors over N_u subcarriers. When a data vector transmitted through (or assigned to) subcarrier i is lost due to channel error (or discarded at the transmitter), it contributes a distortion D_i, given by (2). The mean square distortion in a transmission scheme can be obtained as

$$D = \sum_{i \in N_u} D_i \Psi_i + \sum_{i \notin N_u} D_i, \tag{3}$$

where Ψ_i is the data vector error rate (VER) of subcarrier i, which can be expressed in terms of BER ε_i as

$$\Psi_i = 1 - (1 - \varepsilon_i)^{N'}. \tag{4}$$

A. Scheme I

Here, the total transmit power is minimized via an optimization problem subject to total rate and distortion constraints. It has two arguments n and p_i, and it can be expressed as:

$$\begin{aligned} &\underset{n}{\text{minimize}} \quad p_n^I = \sum_{i \in N_u} p_i \\ &\text{subject to} \quad nr \geq R; \quad \left(\sum_{i \in N_u} D_i \Psi_i + \sum_{i \notin N_u} D_i \right) \leq D^0 \end{aligned} \tag{5}$$

where Ψ_i in (4) can be approximated as: $\Psi_i \approx N' \varepsilon_i$ for $\varepsilon_i \ll 1$. In an uncoded quadrature amplitude modulation (QAM), from [15], the power, rate, and BER are related as:

$$r = \log_2 \left(1 + \frac{-1.5}{\ln(5\varepsilon_i)} p_i g_i \right) \tag{6}$$

Hence, the VER in terms of power, can be expressed as

$$\Psi_i = 0.2N' \exp\left(\frac{-1.5 p_i g_i}{2^r - 1} \right) \tag{7}$$

This problem is combinatorial in nature, due to an integer n. To make it tractable, first we assume the set of used subcarriers N_u is known. For a fixed n, the problem is convex in p_i

because the objective function in (5) is affine and the distortion constraint is convex and can be solved by using Karush-Kuhn-Tucker (KKT) conditions [16].

The Lagrangian function of (5) can be written as:

$$\pounds(\lambda,\{p_i\}) = \sum_{i\in N_u} p_i + \lambda\Bigg(\sum_{i\in N_u} D_i\Psi_i + \sum_{i\notin N_u} D_i - D^0\Bigg) \quad (8)$$

where λ is the Lagrange multiplier. Setting the first derivative of (8) equal to zero, the optimum values are obtained as:

$$p_i = \frac{c_n}{g_i}\ln\left(\frac{N' g_i D_i}{5\mu_n}\right); \quad \varepsilon_i = \frac{\mu_n}{N' g_i D_i} \quad (9)$$

$c_n = \frac{2^{\frac{R}{n}}-1}{1.5}$ and $\mu_n = \frac{c_n}{\lambda}$. λ is calculated when the distortion constraint in (5) holds with equality. Then, μ_n is obtained as:

$$\mu_n = \Bigg(D^0 - \sum_{n\notin N_u} D_i\Bigg) \Big/ \sum_{n\in N_u} \frac{1}{g_i} \quad (10)$$

We consider the optimal selection of N_u and show the convexity of p_n^I with cardinality n. First we study the nature of μ_n. Because of the decreasing importance levels of x_is, D_i is monotonically decreasing. As a result, $D^0 - \sum_{i=n+1}^{N} D_i$ is monotonically increasing with saturation at higher values of n. Note that, the distortion bound D^0 puts a limit on the minimum value n_{min}^I of n. n_{min}^I can be found as $n_{min}^I = \underset{n}{\operatorname{argmin}}\Bigg(\sum\limits_{i=n+1}^{N} D_i \le D^0\Bigg)$. On the other hand, because of sorted channel gains, $\frac{1}{g_i}$ monotonically increases. So, $\sum_{i=1}^{N}\frac{1}{g_i}$ is an exponentially increasing convex sequence. Because of saturating concave nature of the numerator, its the increment rate is higher than that of the denominator at lower values of n, but the trend reverses as n increases. This causes μ_n to increase initially and then reduce with n. Thus, μ_n has a concave nature. Intuitively, $D^0 - \sum_{i=n+1}^{N} D_i \approx 0$ at $n = n_{min}^I$. Also, due to finite positive g_i and exponential growth of $\sum_{i=1}^{N}\frac{1}{g_i}$, at $n > n_{min}^I$ the value of the denominator of μ_n is higher than the numerator, and hence $\mu_n < 1$.

Now, the first difference of p_n^I is defined as $\Delta_{p_n} \triangleq p_n^I - p_{n+1}^I \overset{\text{from (9)}}{=} (c_n - c_{n+1})\sum_{i=1}^{n}\frac{1}{g_i}\ln\frac{N' g_i D_i}{5} - (c_n \ln\mu_n - c_{n+1}\ln\mu_{n+1})\sum_{i=1}^{n}\frac{1}{g_i} - \frac{c_{n+1}}{g_{n+1}}\ln\frac{N' g_{n+1} D_{n+1}}{5\mu_{n+1}}$, having three additive terms. Consider the first two terms. Since c_n is a monotonically decreasing positive convex sequence in n, $c_n - c_{n+1}$ is also positive monotonically decreasing. Further, since μ_n is concave and < 1, $\ln\mu_n$ is a negative valued concave, and hence $c_n\ln\mu_n$ is a concave sequence. So, $c_n\ln\mu_n - c_{n+1}\ln\mu_{n+1}$ is monotonically increasing. The summations $\sum_{i=1}^{n}\frac{1}{g_i}\ln\frac{N' g_i D_i}{5}$ and $\sum_{i=1}^{n}\frac{1}{g_i}$ – both are increasing sequences, but the first sum increases at a slower rate because of the logarithmic component with a decreasing g_i. Thus, due to decreasing $(c_n - c_{n+1})$ and slowing growing $\sum_{i=1}^{n}\frac{1}{g_i}\ln\frac{N' g_i D_i}{5}$ in the first term versus increasing $(c_n\ln\mu_n - c_{n+1}\ln\mu_{n+1})$ and relatively fast growing $\sum_{i=1}^{n}\frac{1}{g_i}$ in the second (negative) term, these two terms combined has a decreasing trend with increase in n. The third term in Δ_{p_n} is the power allocated to the newly added subcarrier, which keeps increasing with n due to reduced g_{n+1}. So, overall, Δ_{p_n} is monotonically decreasing, implying the convex nature of p_n^I.

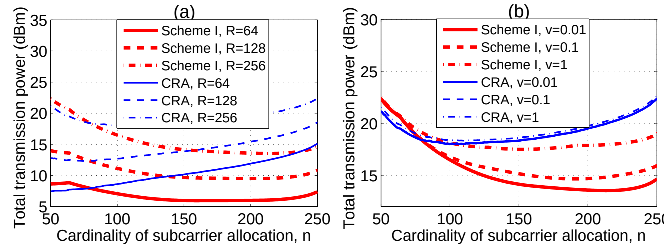

Fig. 1. Total transmit power versus cardinality. (a) $v = 0.01$ (b) $R = 256$ bits/OFDM symbol. Mean square distortion bound $D^0 = 21.8$ dB.

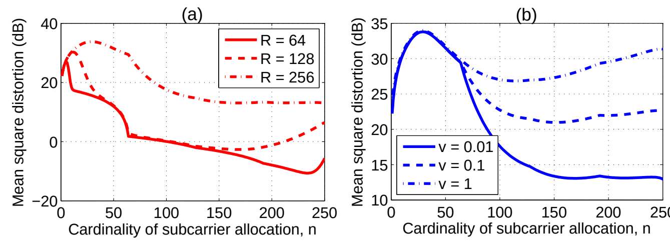

Fig. 2. Variation of mean square distortion for Scheme II. (a) $v = 0.01$ (b) $R = 256$ bits/OFDM symbol. Total transmit power budget $P^0 = 17.8$ dBm.

Figs. 1(a) and (b) show the total transmit power versus cardinality n, where a convex relation can indeed be noted. This numerical algorithm uses golden selection search [17] with the fastest possible mean convergence time to find the optimum n for the minimum of power allocation curve, and it has the same complexity $O(N\log(N))$ as in the CRA scheme. Though the search complexity order is the same as in CRA, the search space in Scheme I is reduced to a range n_{min}^I to N (from 1 to N in CRA) thereby reducing the run time.

Fig. 1(a) also shows that, to maintain a reception quality, total transmit power increases with the demanded rate. Fig. 1(b) illustrates the source dependency on transmit power. A source with a lower v consumes lesser power, even if it requires slightly more number of subcarriers. This is because, a higher gradient in the importance levels of data vectors (marked by a lower v) provides a greater scope for optimization.

B. Scheme II

In scheme II, the power is allocated such that the average distortion is minimized while using a given total amount of power. The optimization problem is formulated as:

$$\begin{aligned} &\underset{n}{\text{minimize}} \quad \Bigg(\sum_{i\in N_u} D_i\Psi_i + \sum_{i\notin N_u} D_i\Bigg) \\ &\text{subject to} \quad nr \ge R; \quad \sum_{i\in N_u} p_i \le P^0 \end{aligned} \quad (11)$$

which can be solved similarly as in Scheme I. The optimization problem (11) in Scheme II provides solutions as in (9), with

$$\mu_n = \exp\left(\frac{\sum_{i\in N_u}\frac{c_n}{g_i}\ln\left(\frac{N' g_i D_i}{5}\right) - P^0}{\sum_{i\in N_u} c_n/g_i}\right) \quad (12)$$

The mean square distortion D^{II} for Scheme II can be obtained from (3) using (7) (VER and allocated power relation).

Fig. 2 shows the variation of mean square distortion with cardinality n. Fig. 2(a) shows that, with increasing rate demand

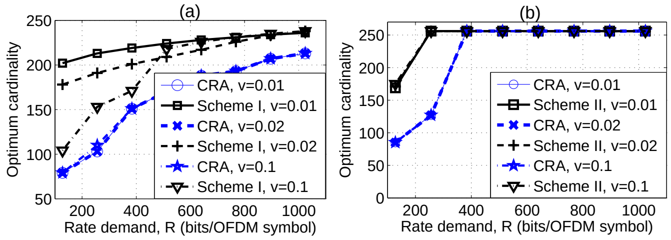

Fig. 3. Optimum cardinality plots: (a) $D^0 = 21.8$ dB. (b) $P^0 = 17.8$ dBm.

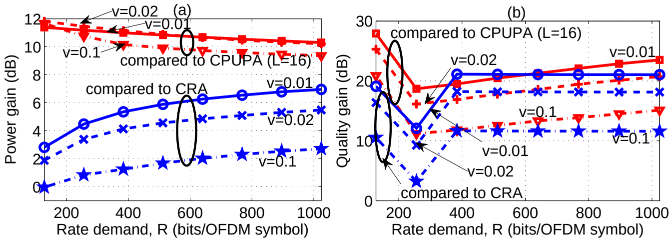

Fig. 4. (a) Power gain in Scheme I, and (b) Quality gain in Scheme II.

R, Scheme II initially reduces the cardinality n so as to increase the available power budget per used subcarrier to control total distortion. But, despite this, total distortion keep rising. Consequently, at higher rates, Scheme II uses all subcarriers to reduce BER per subcarrier. Fig. 2(b) demonstrates the awareness of the Scheme II to the source application. The sources having a lower v provide more flexibility to the optimization problem in adjusting different BERs over a larger set of subcarriers to achieve a higher performance gain.

IV. Results and Discussion

The system performance was studied in MATLAB with $N = 256$ OFDM subcarriers and data vector size $N' = 512$ bits. The average BER was upper bounded at $\varepsilon = 10^{-3}$. Channel gain for each subcarrier was normalized to unity and the noise power was maintained at 10 dB below the signal power. The demanded rate R was varied from 128 to 1024 bits/OFDM symbol, which corresponds to average 0.5 to 4 bits/subcarrier/OFDM symbol. The results include the case of $v = 0.02$, as is the best match for the standard Lena image.

Algorithms proposed in [6] (CRA) and [8] (CPUPA) have been used for comparison with the proposed schemes. To compare power efficiency, CRA scheme has been adapted to adjust the average BER so the total distortion is kept bounded. Also, a power minimization problem has been formulated and solved based on the distortion minimization approach in [8].

Fig. 3(a) demonstrates source dependency of the proposed schemes on the optimal cardinality n. The benefit of a higher source variance (i.e., lower value of v) is extracted more in Scheme I by choosing a higher cardinality and a lesser total power (cf. Fig. 1). Power allocation constraint in Scheme II does not offer much variation in optimum cardinality, although a smaller v offers a lesser distortion (also cf. Fig. 2(b)). The CRA scheme shows very little source dependency. CPUPA has not been considered, because it uses all subcarriers.

Power gains in Scheme I compared to CRA and CPUPA at various source parameter value v are shown in Fig. 4(a). CPUPA is source-aware, while CRA is insensitive to it. As a result, the gain in Scheme I relative to CPUPA at different v is a little lesser than that compared to CRA. However, using all subcarriers in CPUPA causes it to perform worse than the cardinality based approaches (CRA and Scheme I), although the power gain decreases with R because of an increasingly higher cardinality requirement in Scheme I.

Fig. 4(b) shows the quality gain of Scheme II at various R and v values. With increasing R, per-subcarrier BER increases sharply in Scheme II – even after increasing the cardinality n to the maximum, which causes the distortion to rise and hence reduce gain. The uneven optimum n in Scheme II is visible in Fig. 2(a). At further higher rates, the optimum n in Scheme II (as well as in CRA) is saturated (cf. Fig. 3) and the respective distortion performances are equivalent to minimizing $\sum_{i \in N_u} D_i \Psi_i$ and minimizing $\sum_{i \in N_u} \varepsilon_i$, which result in a constant gain. Compared to CPUPA, gain with Scheme II increases with R, indicating the benefit of individual subcarrier level power allocation versus group level granularity.

Thus, the benefit of utilizing source-aware power allocation schemes over OFDM channels is apparent, specially for the sources having a higher degree in importance level variation. The proposed source-aware techniques can be extended to variable rate applications, such as video/multimedia streaming.

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