CHAPTER 6

The Heat Equation

We introduce several PDE techniques in the context of the heat equation:

The Fundamental Solution is the heart of the theory of infinite domain problems. The fundamental solution also has to do with bounded domains, when we introduce Green's functions later.

The Maximum Principle applies to the heat equation in domains bounded in space and time. It is an important property of parabolic equations used to deduce a variety of results such as uniqueness of solutions, comparison principles. (E.g., if boundary conditions are changed in a way that suggests intuitively the resulting temperature should be smaller, this can be proved using the maximum principle.)

The Energy Method works analogously to the wave equation, except that the physical (heat) energy is less interesting than a mathematical energy, which typically decays. As for the wave equation, this leads to straightforward uniqueness results. It is also useful for obtaining estimates on solutions that are part of the existence and regularity theory for parabolic equations.

Initial Boundary Value Problems. We will spend some time describing explicit solutions, expressed as infinite series of functions, of the heat equation plus initial and boundary conditions. There is a general technique frequently referred to as separation of variables, or as eigenfunction expansions. The development of this technique leads us to an analysis of eigenvalue problems for ordinary and partial differential equations, and to the analysis of Fourier series.

6.1. The Fundamental Solution

To start with, we consider the heat equation in one space variable, plus time. In this section, we derive the fundamental solution and show how it is used to solve the *Cauchy problem:*

$$\begin{aligned} u_t &= k u_{xx}, \qquad |x| < \infty, t > 0 \\ u(x,0) &= g(x), \qquad |x| < \infty \end{aligned}$$

The heat equation has a scale invariance property that is analogous to scale invariance of the wave equation or scalar conservation laws, but the scaling is different.

Let $a > 0$ be a constant. Under the scaling $x \to ax, \quad t \to a^2 t$ the heat equation is unchanged. More precisely, if we introduce the change of variables: $\bar{t} = a^2 t, \ \overline{x} = ax$, then the heat equation becomes

$$u_{\bar{t}} = k u_{\overline{x}\overline{x}}$$

This scale invariance suggests that we seek solutions v depending on the similarity variable $\frac{x^2}{t}$, or on $\frac{x}{\sqrt{t}}$. However, there is a property of the heat equation we would like to preserve in our similarity solution, that of conservation of energy. Suppose u is a solution of the heat equation with the property that $|\int_{-\infty}^{\infty} u(x,0)\,dx| < \infty$, and $u_x(x.t) \to 0$ as $x \to \pm\infty$. Then, integrating the PDE, we find

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\,dx = 0,$$

so that the total heat energy is conserved:

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \text{constant}. \tag{1.1}$$

However,

$$\int_{-\infty}^{\infty} v\left(\frac{x}{\sqrt{t}}\right)\,dx = t^{\frac{1}{2}}\int_{-\infty}^{\infty} v(y)\,dy.$$

This suggests we should scale the function v by $t^{-\frac{1}{2}}$:

$$u(x,t) = \frac{1}{\sqrt{t}}\, v\left(\frac{x}{\sqrt{t}}\right) \tag{1.2}$$

With this scaling, heat is conserved in the sense of (1.1).

Substituting (1.2) into the PDE leads to an ODE for $v = v(y)$, with non-constant coefficients:

$$k v''(y) + \frac{1}{2} y v'(y) + \frac{1}{2} v(y) = 0 \tag{1.3}$$

Since this is a second order equation, we should have two independent solutions. First rewrite the ODE as

$$k v''(y) + \frac{1}{2}(y v(y))' = 0.$$

Thus,

$$k v'(y) + \frac{1}{2} y v(y) = \text{ constant}.$$

Since we are really only seeking one solution, it is convenient to set the constant to zero, and write the solution of the homogenous equation:

$$v(y) = Ae^{-\frac{y^2}{4k}}.$$

Converting back to (x, t), with $y = \frac{x}{\sqrt{t}}$, we obtain the similarity solution

$$u(x,t) = A\frac{1}{\sqrt{t}}\, e^{-\frac{x^2}{4kt}} \tag{1.4}$$

Usually, we choose a particular value of A so that the constant in (1.1) is unity; for this choice of constant, we have the *fundamental solution* of the heat equation:

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi kt}}\, e^{-\dfrac{x^2}{4kt}}.$$

In higher dimensions, $x \in \mathbb{R}^n$, the fundamental solution takes a similar form:

$$\begin{aligned} \Phi(\mathbf{x},t) &= \frac{1}{(4\pi kt)^{n/2}} e^{-|\mathbf{x}|^2/4kt} \\ &= \phi(r,t), \qquad r = |x|, \end{aligned}$$

where $\phi(r,t) = \frac{1}{(4\pi kt)^{n/2}} e^{-r^2/4kt}$. Then $u(\mathbf{x},t) = \Phi(\mathbf{x},t)$ satisfies

$$u_t = k\Delta u, \qquad \Delta = \text{ div grad} \quad (\text{the } \textit{Laplacian} \text{ with respect to } x).$$

Exercise. Show that $\Delta\phi(r,t) = \phi_{rr} + \frac{n-1}{r}\phi_r$. Hence,

$$\phi_t = k\left(\phi_{rr} + \frac{n-1}{r}\phi_r\right).$$

Properties of the Fundamental Solution $\Phi(x,t)$

1. $\Phi(x,t) > 0$ for all $x \in \mathbb{R}, t > 0$
2. Φ is C^∞ in (x,t), $t > 0$
3. $\int_{\mathbb{R}^n} \Phi(x,t)dx = 1$ for all $t > 0$.

Properties 1 and 2 are obvious; here is the proof of property 3 for $n = 1$.

$$\int_{-\infty}^{\infty} \Phi(x,t)dx = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4kt}}\, dx$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy \qquad (\text{let } y = \frac{x}{\sqrt{4kt}}, \quad dy = \frac{dx}{\sqrt{4kt}})$$
$$= 1.$$

■

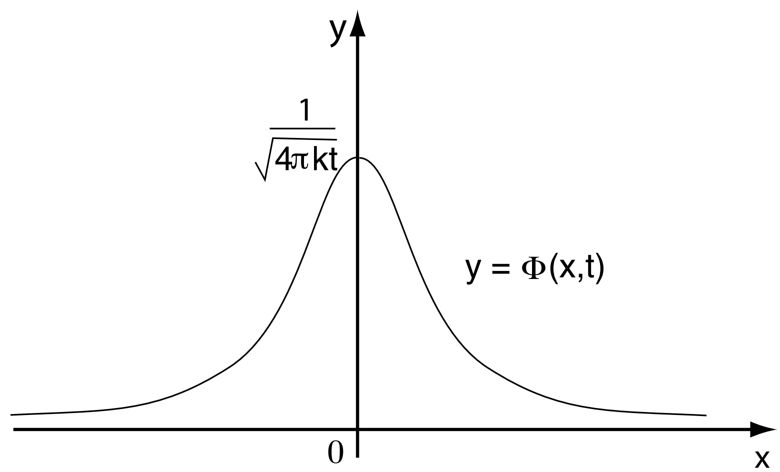


FIGURE 6.1. Graph of the fundamental solution for the heat equation, with $t > 0$. .

Therefore, $\Phi(x,t)$ is a probability distribution for each $t > 0$, with interesting dependence on t in the limits $t \to \infty$ and $t \to 0$:

The area under the graph is 1 for all $t > 0$, yet as $t \to \infty$, $\max_x \Phi(x,t) \to 0$; the tail spreads out to maintain $\int \Phi = 1$

As $t \to 0$ the maximum (at $x = 0$) blows up like $\frac{1}{\sqrt{t}}$, but the integral remains constant. We also observe $\Phi(x,t) \to 0$ for $x \neq 0$, as $t \to 0+$.

6.2. The Cauchy Problem for the Heat Equation

As for the wave equation, the Cauchy problem is the pure initial value problem, here stated in the one-dimensional case, $n = 1$:

$$(2.5) \qquad \begin{aligned} u_t &= k u_{xx}, \qquad |x| < \infty,\ t > 0 \qquad &(a) \\ u(x,0) &= g(x), \qquad |x| < \infty \qquad &(b) \end{aligned}$$

Recall that the fundamental solution

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

satisfies (2.5(a)) for $t > 0$.

Now $\Phi(x-y,t)$ is a solution of (2.5(a)) for all y, by translation invariance: $x \to x - y$ does not change the heat equation. Thus,

$$\Phi(x-y,t)g(y)$$

is also a solution of (2.5(a)). For later reference, we note that the heat equation is invariant under time translation also.

By linearity and homogeneity of the PDE, we can also take linear combinations of solutions. This suggests that

$$u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t)g(y)dy \tag{2.6}$$

should also be a solution. Moreover, properties of Φ suggest that as $t \to 0+$, $u(x,t) \to g(x)$, since $\Phi(x-y,t)$ collapses to zero away from $y = x$, and blows up at $y = x$ in such a way (i.e., preserving $\int \Phi = 1$) that the initial condition is satisfied in the sense $u(x,t) \to g(x)$ as $t \to 0+$.

It is straightforward to check that the integrals for u, u_t, u_{xx} all converge provided $g \in C(\mathbb{R})$ is bounded. Then

$$u_t = \int_{-\infty}^{\infty} \frac{\partial \Phi}{\partial t}(x-y,t)g(y)dy; \qquad u_{xx} = \int_{-\infty}^{\infty} \frac{\partial^2 \Phi}{\partial x^2}(x-y,t)g(y)dy,$$

so that u satisfies the PDE for $t > 0$.

It is more complicated to check that the initial condition is satisfied. We need to show $u(x,0) = g(x)$. But $t = 0$ is a singular point for $\Phi : \Phi(x,t)$ is not defined at $t = 0$. To get an idea of why $\lim_{t\to 0+} u(x,t) = g(x)$, let's fix x.

Then, for $\delta > 0$,

$$\begin{aligned} &\textstyle\int_{-\infty}^{\infty} \Phi(x-y,t)g(y)dy \\ &= \textstyle\int_{(x-y)<\delta} \Phi(x-y,t)g(y)dy + \int_{|x-y|>\delta} \Phi(x-y,t)g(y)dy \\ &\approx \textstyle\int_{|x-y|<\delta} \Phi(x-y,t)g(x)dy \end{aligned}$$

By continuity, $g(y) \approx g(x)$ for y near x, which explains how the first integral is approximately the final line. The second integral approaches zero as $t \to 0+$, because $\Phi \to 0$ uniformly, and exponentially, away from $y = x$ as $t \to 0+$.

Theorem: Let $g \in C(\mathbb{R})$ be bounded, and let $u(x,t)$ be given be the formula (2.6). Then

1. u is C^∞ in (x,t) for $t > 0$;

2. u satisfies the heat equation

$$u_t = k u_{xx}, \quad x \in \mathbb{R},\ t > 0;$$

3. $$\lim_{\substack{(x,t) \to (x_0, 0) \\ t > 0}} u(x,t) = g(x_0) \quad \text{for all} \quad x_0 \in \mathbb{R}.$$

Proof:

Property 1 follows because Φ is C^∞ for $t > 0$, and derivatives of Φ all decay exponentially as $|x| \to \infty$, so the integrals converge.

Property 2 follows from $\Phi_t = k\Phi_{xx}, \quad t > 0$.

To prove property 3, we look at the difference $|u(x,t) - g(x_0)|$, estimate and show that the pieces we get behave as we expect, i.e., as in the rough argument preceeding the proof.

Let $\epsilon > 0$. (This measures $|u(x,t) - g(x_0)|$.) Then, since $\int \Phi\, dx = 1$, we have

$$|u(x,t) - g(x_0)| = \left| \int_{-\infty}^{\infty} \Phi(x-y,t)(g(y) - g(x_0))\, dy \right|, \tag{2.7}$$

Let $\delta > 0$ (we choose δ below), and break up the integrals in (2.7):

$$|u(x,t) - g(x_0)| \leq \left| \int_{|x_0 - y| < \delta} \Phi(x-y,t)(g(y) - g(x_0)) dy \right| - \left| \int_{|x_0 - y| \geq \delta} \Phi(x-y,t)(g(y) - g(y_0)) dy \right| \tag{2.8}$$

Now we use δ two ways to show the two integrals are small:

1st: $g(y) \sim g(x_0)$. For y near x, $\Phi(x-y,t)$ blows up as $t \to 0$, but $\int \Phi(x-y,t)\, dy$ is bounded uniformly by 1.

2nd: $\int_{|x_0 - y| \geq \delta} \Phi(x-y,t)\, dy \to 0$ as $t \to 0$, provided $|x - x_0| < \frac{\delta}{2}$, while $g(y)$ is bounded.

We write the right hand side of (2.8) as

$$I_\delta + J_\delta$$

Choose $\delta > 0$ so that $|g(y) - g(x_0)| < \epsilon$ for $|y - x_0| < \delta$ (by continuity of g at x_0).

Then

$$\begin{aligned} I_\delta &\le \int_{|x_0-y|<\delta} \Phi(x-y,t)|g(y)-g(x_0)|dy \\ &\le \epsilon \int_{|x_0-y|<\delta} \Phi(x-y,t)dy \ \le \ \epsilon. \end{aligned}$$

The second integral is somewhat trickier: Since g is bounded, there is $K > 0$ such that $|g(y)| \le K$, for all y. Thus,

$$\begin{aligned} J_\delta &\le \int_{|x_0-y|>\delta} \Phi(x-y,t)|g(y)-g(x_0)|dy \\ &\le 2K \int_{|x_0-y|>\delta} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} \ dy \end{aligned}$$

Here is the second use of δ : Consider x satisfying $|x - x_0| < \frac{\delta}{2}$. Then $|x-y| > \frac{\delta}{2}$ in the range of integration $|x_0 - y| > \delta$. But this is not a good enough estimate of the exponential, because we would still be left with an integral over an infinite interval of a small but positive quantity. So, we need to observe that in the region of integration, $|x-y| \ge \frac{1}{2}|y-x_0|$. Then

$$\begin{aligned} J_\delta &\le \tfrac{K}{\sqrt{\pi k}} \textstyle\int_{|x_0-y|>\delta} \frac{1}{\sqrt{t}} e^{-\frac{(x_0-y)^2}{16kt}} \ dy \\ &\le C \textstyle\int_{|z|\ge \frac{\delta}{4\sqrt{kt}}} e^{-z^2} \ dz \ < \epsilon, \end{aligned}$$

for $t > 0$ sufficiently small.

Thus, $|u(x,t) - g(x_0)| < 2\epsilon$ for $|x - x_0| < \frac{\delta}{2}$, $t > 0$ sufficiently small.

This proves property 3. ■

Remarks

1. $\lim_{t\to 0+} \Phi(x,t)$ is not a function in the usual sense, but is a *distribution* or *generalized function* called the *Dirac delta function* $\delta(x)$:

$$\int_{\mathbb{R}} \delta(x-y)g(y) \ dy = g(x),$$

where the integral is a notational convenience. One way to interpret the delta function and the integral is to recognize the delta function as a measure that places unit mass at $x = 0$ and zero mass at each $x \neq 0$.

2. $\int_{-\infty}^{\infty} \Phi(x-y,t) \ g(y) \, dy$ is the **convolution** of the function $\Phi(\cdot,t)$ with g More generally, for integrable functions ϕ, ψ on $\mathbb{R}$ the convolution product of ϕ and ψ is also a function $\phi * \psi$ defined by

$$\phi * \psi(x) = \int_{-\infty}^{\infty} \phi(x-y)\psi(y) \ dy.$$

6.2.1. Using the Fundamental Solution to Solve Quarter-plane Problems. :

Consider the so-called *quarter-plane* problem:

1) $u_t = ku_{xx},\ x > 0,\ t > 0$

2) $u(0,t) = 0,\ t > 0.$ (Homogeneous Dirichlet boundary condition.)

3) $u(x,0) = g(x),\ x > 0$

As for the wave equation, we reflect the initial data so the solution satisfies the boundary conditions:

Let $\tilde{g}(x)$ be the odd extension of $g(x)$.

$$\tilde{g}(x) = \begin{cases} g(x), & x > 0 \\ -g(-x), & x < 0 \end{cases}$$

and define

$$u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t)\tilde{g}(y)dy \tag{2.9}$$

Then

$$\begin{aligned} u_t &= ku_{xx}, \\ u(x,0) &= \tilde{g}(x) \ = \ g(x), \quad x > 0 \\ u(0,t) &= \int_{-\infty}^{\infty} \Phi(-y,t)\tilde{g}(y)dy \ = \ 0, \quad t > 0, \end{aligned}$$

since $\Phi(y,t)$ is an even function of y, and $\tilde{g}$ is an odd function.

It is straightforward to check directly that $u(x,t)$ is an odd function of $x \in \mathbb{R}$. That is, the symmetry in the initial data is carried through to the same symmetry in the solution.

Now replace $\tilde{g}(y)$ with $g(y)$ using $\tilde{g}(y) = -g(-y), \quad y < 0$. Then

$$\begin{aligned} u(x,t) &= \int_0^{\infty} \Phi(x-y,t)g(y)dy + \int_{-\infty}^{0} \Phi(x-y,t)(-g(-y))dy \\ &= \int_0^{\infty} \Phi(x-y,t)g(y)dy - \int_0^{\infty} \Phi(x+y,t)g(y)dy \\ &= \int_0^{\infty} (\Phi(x-y,t) - \Phi(x+y,t))g(y)dy \end{aligned}$$

With a homogeneous Neumann boundary condition instead, we extend the initial data to be even:

$$\begin{aligned} u_t &= k u_{xx}, \quad x > 0, \quad t > 0 \\ u_x(0,t) &= 0 \qquad\qquad \text{(Homogeneous Neumann Condition)} \\ u(x,0) &= g(x), \quad x > 0 \end{aligned}$$

Solution: Extend g using the even extension, so that the first derivative is zero at $x = 0$.

Then $\quad u(x,t) = \int_0^\infty (\Phi(x-y,t) + \Phi(x+y,t))\, g(y)\, dy.$

To check the boundary condition:

$u_x(0,t) = \int_0^\infty (\Phi_x(-y,t) + \Phi_x(y,t)\, g(y)\, dy.$

$\Phi(y,t)$ is even in y so $\Phi_x(y,t)$ is odd. Thus, $u_x(0,t) = 0$.

6.3. The Energy Method.

Consider

$$u_t = k\ u_{xx}, \qquad a \leq x \leq b, t > 0.$$

multiplying by u and integrating over $[a,b]$,

$$\int_a^b u u_t\ dx = k \int_a^b u u_{xx} dx.$$

Therefore, $$\frac{d}{dt}\int_a^b \frac{1}{2}u^2 dx = k u u_x|_a^b - k\int_a^b u_x^2 dx.$$

Thus, the energy integral (not the heat energy) $E(t) = \int_a^b \frac{1}{2}u^2(x,t)dx$ is decreasing in time if $u u_x|_a^b \leq 0$.

For example, if either u or u_x is zero at each end point $x = a, x = b$, then the energy integral decreases in t.

In this case, we have the important comparison to the initial data:

$$E(t) \leq E(0), \qquad t > 0.$$

I.e.,

$$\int_a^b u^2(x,t)dx \leq \int_a^b u^2(x,0)dx, \qquad t > 0. \tag{3.10}$$

6.3.1. Using the Energy Method to prove Uniqueness. Consider the initial boundary value problem

$$\text{(IBVP)} \quad \begin{cases} u_t = ku_{xx} + f(x,t), & 0 < x < l, \quad t > 0 \\ \\ u(0,t) = g(t), & t > 0 \\ u(L,t) = h(t), & t > 0 \\ \\ u(x,0) = \phi(x), & 0 < x < L \end{cases}$$

where f, g, h, ϕ are given functions.

The energy inequality works in any number of dimensions (will involve divergence theorem and Green's identities instead of integration by parts).

Theorem. If u_1, u_2 solve (IBVP) and are C^2 functions, then $u_1 = u_2$ everywhere.

Proof: Let $u = u_1 - u_2$

Then u satisfies (IBVP) with zero data: $f \equiv g \equiv h \equiv \phi \equiv 0$. But we know that $E(t)$ is decreasing, from the energy inequality

$$0 \leq \int_0^L (u(x,t))^2 dx \quad \leq \quad \int_0^L u(x,0)^2 \; dx = 0$$

Therefore, $u \equiv 0$, so that $u_1 \equiv u_2$.

Remark. In general, the existence of solutions is harder to establish than uniqueness.

6.3.2. The Energy Principle in Higher Dimensions: Consider a bounded open subset U of $\mathbb{R}^n$, and the initial boundary value problem

$$\begin{aligned} u_t &= k\Delta u, && x \in U \quad t > 0 \\ u(x,t) &= 0, && x \in \partial U, \quad t > 0 \\ u(x,0) &= \phi(x), && x \in U \end{aligned}$$

We begin by defining the energy integral as in one dimension:

$$E(t) = \frac{1}{2} \int_U u(x,t)^2 \; dx. \tag{3.11}$$

Then

$$\begin{aligned} E'(t) &= \int_U uu_t \, dx = k\int_U u\Delta u \, dx \\ &= k\int_{\partial u} u\nabla u \cdot n \, dS - k\int_U \nabla u \cdot \nabla u \, dx \\ &= -k \textstyle\int_U |\nabla u|^2 dx \leq 0. \end{aligned}$$

Exercise: Prove the energy inequality for $u_t = \nabla.(k(x,u)\nabla u)$ where $k(x,u) \in \mathbb{R}^n$ is a given positive function. The only difference is that k gets moved under the integral sign. Does this enable us to prove uniqueness of solutions for the quasilinear heat equation?

6.4. The Maximum Principle

For $T > 0$, and $U \subset \mathbb{R}^n$, we use the notation $U_T = U \times (0, T]$. Note that U_T includes the *top* $U \times \{t = T\}$. The following theorem, the **Maximum Principle,** states that the maximum of any (smooth) solution of the heat equation occurs either initially (at $t = 0$), or on the boundary of the domain. These parts of the boundary of U_T are known as the *parabolic boundary* Γ_T :

$$\Gamma_T = \overline{U}_T - U_T = (\partial U \times [0, T]) \cup (U \times \{t = 0\}).$$

Solutions of the heat equation should have two spatial derivatives and one time derivative, so we define the appropriate space of functions on U_T:

$$C_1^2(U_T) = \{u = u(x,t) : u, u_t, D_x u, D_x^2 u \in C(U_T)\}.$$

In order to compare values of u in U_T with values on the boundary, we require in the theorem that u should be continuous on $\overline{U}_T$.

Theorem. The Maximum Principle. Let $u \in C(\overline{U_T}) \cap C_1^2(U_T)$ satisfy

$$u_t = k\Delta u, \qquad (x,t) \in U_T.$$

Then

$$\max_{\overline{U}_T} u(x,t) = \max_{\Gamma_T} u(x,t).$$

<u>Remarks</u>:

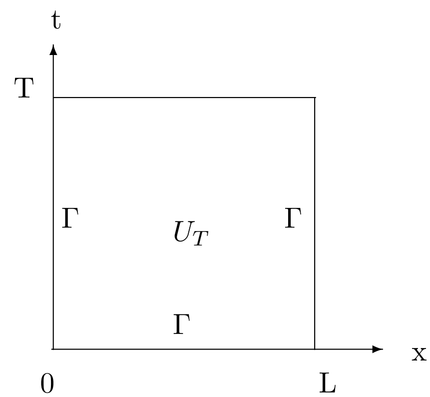
1. Suppose u has a local maximum at $(x,t) \in U_T$. Then $u_t = 0 = u_x = 0$, $\Delta u \leq 0$. If $\Delta u < 0$ at (x,t), then the pde gives us a contradiction:

$$0 = u_t \;=\; k\Delta u < 0$$

Although this is not a proof, since we have to handle the degenerate case in which $\Delta u = 0$, it has the main idea; the proof merely modifies u to remove a possibly degenerate maximum.

2. $\overline{U}_T$ is closed, so u achieves it's maximum somewhere in $\overline{U}_T$: there is an (x_0, t_0) such that $u(x_0, t_0) = \max_{\overline{U}_T} u(x, t)$.

<u>Proof</u>: Let $M = \max_{\Gamma_T} u(x, t)$



<u>Goal</u>: Prove that $U(x, t) \leq M$ for all $(x, t) \in U_T$

To deal with the possibility $\Delta u = 0$ at a maximum, we perturb u a bit:

Let $v(x, t) = u(x, t) + \epsilon |x|^2, \quad \epsilon > 0$. Then

$$v_t - k\Delta v = \underbrace{u_t - k\Delta u}_{0} - 2\,k\,n\,\epsilon < 0 \tag{4.12}$$

Now suppose v has a local maximum in U_T, at $P_0 = (x_0, t_0)$ with $t_0 < T$. Then $v_t = v_x = 0$ at P_0, and $\Delta v \leq 0$ at P_0. But this contradicts (4.12), so v cannot have a max in the interior of U_T.

Now suppose v has a maximum on the line $t = T$, at $P_1 = (x_1, t = T)$. Then $v_x = 0$, $\Delta v_{xx} \leq 0$ and $v_t \geq 0$ at P_1. [1] Now we have $v_t - k\Delta v \geq 0$, again contradicting (4.12).

[1]The final inequality is easily proved by contradiction - $v_t < 0$ at P_1, then $v(x_1, t) > v(x_1, T)$ for $t < T$ close to $t - T$, contradicting the assumption that v has a maximum at P_1.

Therefore maximum of v on $\overline{U}_T$ occurs on Γ_T. I.e., $v(x,t) \leq \max_{(y,t)\in\overline{U}_T} v(y,t)$ for all $(x,t) \in \overline{U}_T$. We have proved

$$u(x,t) + \epsilon|x|^2 \leq \max_{(y,t)\in\Gamma_T} (u(y,t) + \epsilon|y|^2) \leq \max_{\Gamma_T} u + \epsilon C \qquad \text{for all } (x,t) \in \overline{U}_T,$$

where $C = \max_{y\in U} |y|^2$. Thus,

$$\begin{aligned} u(x,t) &\leq M + \epsilon(C - |x|^2) \\ &\leq M + \epsilon C \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $u(x,t) \leq M$ for all $(x,t) \in \overline{U}_T$.

Remarks:

1. The weak maximum principle is easy to prove. The related *strong maximum principle* is somewhat harder to prove. The strong maximum principle states that, provided U is connected, then the maximum of u is achieved *only* on the parabolic boundary, unless u is constant throughout $\overline{U}_T$.

2. By applying the maximum principle to $-u$, which also satisfies the conditions of the Theorem, we see that there is a corresponding minimum principle:

$$\min_{\overline{U}_T} u(x,t) = \min_{\Gamma_T} u(x,t).$$

6.5. Duhamel's Principle for the Inhomogeneous Heat Equation

Consider the example of the heat equation on the whole real line, with a heat source or sink represented by the density function $f(x,t)$:

$$(\mathrm{P}) \qquad \begin{cases} u_t = ku_{xx} + f(x,t), & |x| < \infty, t > 0 \\ u(x,0) = 0 & |x| < \infty \end{cases}$$

The fundamental solution $\Phi(x,t)$ of the homogeneous heat equation satisfies $u_t = ku_{xx}$, for $t > 0$. But then, for any $y, s > 0$ the shifted function $\Phi(x - y, t - s)$ satisfies the equation for $t > s$, and we can multiply by an *amplitude* $f(y,s)$, so that we have a collection of solutions $\Phi(x-y,t-s)f(y,s)$ of the heat equation, with $y \in \mathbb{R}$ and $t > s$. Summing (i.e., integrating) these solutions over $y \in \mathbb{R}$, with s fixed, we have that

$$\tilde{u}(x,t;s) = \int_{-\infty}^{\infty} \Phi(x-y,t-s)f(y,s)\,dy \qquad \text{is a solution for all } t > s,$$

satisfying $\quad \tilde{u}(x,t=s;s) = f(x,s)$

Now the idea is to integrate $\tilde{u}(x,t;s)$ with respect to s from 0 to t :

Define $u(x,t) = \int_0^t \tilde{u}(x,t,s)ds$. Then

$$\begin{aligned} u_t - ku_{xx} &= \tilde{u}(x,t;t) + \int_0^t (\tilde{u}_t(x,t;s) - k\tilde{u}_{xx}(x,t;s))ds \\ &= f(x,t) \end{aligned}$$

Note that this calculation is slightly misleading, because we have differentiated under the integral sign, when $\Phi(x-y, t-s)$ has a singularity at $x = y, t = s$, on the boundary of the domain of integration. However, this singularity can be handled with the appropriate limit, and the result is the same. (See Evans [PDEs], page 50 for the details.)

Finally, we observe

$$u(x,0) = 0 \quad -\infty < x < \infty.$$