The Legendre Equation

Some of the important differential equations met in reelworld applications are second order linear equations with analytic coefficients. One of these is the Legendre equation

$$L(y) \equiv (1-x^2)\, y'' - 2x\, y' + \alpha(\alpha+1)\, y = 0$$

where α is a constant.

If we write this equation in the form

$$y'' - \frac{2x}{1-x^2}\, y' + \frac{\alpha(\alpha+1)}{1-x^2}\, y = 0$$

we see that the functions a_1, a_2 given by

$$a_1(x) = \frac{-2x}{1-x^2}, \quad a_2(x) = \frac{\alpha(\alpha+1)}{1-x^2}$$

Can be expressed in the power series

$$a_1(x) = -2x \cdot \{1 + x^2 + x^4 + \cdots\}$$

$$a_2(x) = \alpha(\alpha+1)\,\{1 + x^2 + x^4 + \cdots\}.$$

and both converge for $|x| < 1$.

From Theorem 12 it follows that the solutions of $L(y)=0$ on $|x|<1$ have convergent power series expansions there.

Now we proceed to find a basis for these solutions.

Let $\phi(x) = \sum_0^\infty c_k x^k$ be any solution of the Legendre equation on $|x|<1$.

Then
$$\phi'(x) = \sum_0^\infty k c_k x^{k-1},$$
$$-2x\phi'(x) = \sum_0^\infty -2k c_k x^k,$$
$$\phi''(x) = \sum_0^\infty k(k-1) c_k x^{k-2}$$
$$-x^2\phi''(x) = \sum_0^\infty -k(k-1) c_k x^k.$$

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Note that $\phi''(x)$ may also be written as 3

$$\phi''(x) = \sum_{0}^{\infty} (k+2)(k+1) c_{k+2} x^k .$$

Substituting all these into $L(y)=0$,

$$L(\phi)(x) = (1-x^2)\phi''(x) - 2x\phi'(x) + \alpha(\alpha+1)\phi(x)$$

$$= \sum_{0}^{\infty} \Big[(k+2)(k+1) c_{k+2} - k(k-1) c_k - 2k c_k + \alpha(\alpha+1) c_k \Big] x^k$$

$$= \sum_{0}^{\infty} \Big[(k+2)(k+1) c_{k+2} + (\alpha+k+1)(\alpha-k) c_k \Big] x^k$$

For ϕ to satisfy $L(\phi)=0$ we must have all the coefficients of the powers of x equal to zero. Hence

$$(k+2)(k+1) c_{k+2} + (\alpha+k+1)(\alpha-k) c_k = 0$$
$$(k=0,1,2,3,\dots)$$

<u>Put $k=0$:</u>

$$2 \cdot 1 \cdot c_2 + (\alpha+1)(\alpha) \cdot c_0 = 0$$

$$\Rightarrow \quad c_2 = -\frac{\alpha(\alpha+1) c_0}{1 \cdot 2}.$$

<u>For $k=1$:</u>

$$3 \cdot 2 \cdot c_3 + (\alpha+2)(\alpha-1)\, c_1 = 0$$

$$\Rightarrow c_3 = -\frac{(\alpha+2)(\alpha-1)}{3 \cdot 2} c_1$$

Similarly, letting $k = 2, 3$, we obtain

$$c_4 = -\frac{(\alpha+3)(\alpha-2)}{4 \cdot 3} c_2$$

$$= \frac{(\alpha+3)(\alpha+1)\,\alpha\,(\alpha-2)}{4 \cdot 3 \cdot 2} c_0$$

$$c_5 = -\frac{(\alpha+4)(\alpha-3)}{5 \cdot 4} c_3$$

$$= \frac{(\alpha+4)(\alpha+2)(\alpha-1)(\alpha-3)}{5 \cdot 4 \cdot 3 \cdot 2} c_1 .$$

By induction it follows for $m = 1, 2, 3, \ldots$

$$c_{2m} = (-1)^m \cdot \frac{(\alpha+2m-1)(\alpha+2m-3)\cdots(\alpha+1)(\alpha)(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!} c_0$$

$$c_{2m+1} = (-1)^m \frac{(\alpha+2m)(\alpha+2m-2)\cdots(\alpha+2)(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!} c_1$$

All coefficients are determined in terms of c_0 and c_1, and we must have

$$\phi(x) = c_0\,\phi_1(x) + c_1\,\phi_2(x)$$

where

$$\phi_1(x) = 1 - \frac{(\alpha+1)\alpha}{2!}x^2 + \frac{(\alpha+3)(\alpha+1)\alpha(\alpha-2)}{4!}x^4 - \cdots$$

or

$$\phi_1(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{(\alpha+2m-1)(\alpha+2m-3)\cdots(\alpha+1)\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!}x^{2m},$$

and

$$\phi_2(x) = x - \frac{(\alpha+2)(\alpha-1)}{3!}x^3 + \frac{(\alpha+4)(\alpha+2)(\alpha-1)(\alpha-3)}{5!}x^5 - \cdots$$

or

$$\phi_2(x) = x + \sum_{1}^{\infty} (-1)^m \cdot \frac{(\alpha+2m)(\alpha+2m-2)\cdots(\alpha+2)(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!}x^{2m+1}$$

But ϕ_1 and ϕ_2 are solutions of the Legendre equation, those corresponding to the choices

$c_0 = 1,\ c_1 = 0$. and $c_0 = 0,\ c_1 = 1$. respectively.

They form a basis for the solutions, since $\phi_1(0) = 1$, $\phi_2(0) = 0$

$$\phi_1'(0) = 0, \quad \phi_2'(0) = 1.$$

$$W(\phi_1, \phi_2)(0) = 1 \neq 0.$$

We notice that if α is non-negative integer. $n = 2m$, $(m = 0, 1, 2, \dots)$ then ϕ_1 has only a finite number of non-zero terms. Indeed, in this case ϕ_1 is a polynomial of degree n containing only even powers of x.

For example,

$$\phi_1(x) = 1, \quad (\alpha = 0).$$

$$\phi_1(x) = 1 - 3x^2, \quad (\alpha = 2).$$

$$\phi_1(x) = 1 - 10x^2 + \frac{35}{3}x^4, \quad (\alpha = 4)$$

The solution ϕ_2 is not a polynomial in this case since none of the coefficients in the series of $\phi_2(x)$ vanish.

A similar situation occurs when α is a +ve odd integer n. Then ϕ_2 is a polynomial of degree 'n' having only odd powers of x, and ϕ_1 is not a polynomial.

For example,

$$\phi_2(x) = x, \qquad (\alpha = 1).$$

$$\phi_2(x) = x - \frac{5}{3}x^3, \quad (\alpha = 3).$$

$$\phi_2(x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5, (\alpha = 5).$$

We consider in more detail these polynomial solutions when $\alpha = n$, a non-negative integer. The polynomial solution P_n of degree n of

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0,$$

satisfying $P_n(1) = 1$ is called the n^{th} Legendre polynomial. In order to justify this definition we must show that there is just one such solution for each nonnegative integer n. We shall establish in the below:

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Let ϕ be the polynomial of degree n defined by

$$\phi(x) = \frac{d^n}{dx^n}(x^2-1)^n.$$

This ϕ satisfies the Legendre equation:

$$(1-x^2)y'' - 2x\,y' + n(n+1)\,y = 0.$$

let $u(x) = (x^2-1)^n$.

$$\Rightarrow u'(x) = n(x^2-1)^{n-1} \times (2x)$$

$$\Rightarrow (x^2-1)\,u' - 2nx\,u = 0$$

Differentiating this expression $n+1$ times yields,

$$(x^2-1)\,u^{(n+2)} + 2x(n+1)\,u^{(n+1)} + 2\,\tfrac{(n+1)n}{2}\,u^{(n)} - 2n\left\{x\,u^{(n+1)} + (n+1)\,u^{(n)}\right\} = 0.$$

$$(x^2-1)\,u^{(n+2)} + 2x\,u^{(n+1)} - n(n+1)\,u^{(n)} = 0$$

Since $\phi = u^{(n)}$, ϕ satisfies the Legendre equation.

This polynomial ϕ satisfies

$$\phi(1) = 2^n n!$$

This can be proved in the below:

$$\phi(x) = \left[(x^2-1)^n\right]^{(n)}$$

$$= \left[(x-1)^n (x+1)^n\right]^{(n)}$$

$$= \left[(x-1)^n\right]^{(n)} (x+1)^n$$

$$+ n\left[(x-1)^n\right]^{(n-1)} n(x+1)^{n-1}$$

$$+ \cdots + (x-1)^n \left[(x+1)^n\right]^{(n)}.$$

$$= n!\,(x+1)^n +$$ terms with $(x-1)$ as a factor.

Hence

$$\phi(1) = n!\,2^n.$$