1. (a) $\phi''(x+2\pi) + \sin(x+2\pi)\phi'(x+2\pi) + \phi(x+2\pi) = 0$

(2 Marks) $\Rightarrow \psi'' + (\sin x)\psi' + \psi = 0$

$\Rightarrow \psi$ is a solution.

(b) $\underset{\sim}{\Rightarrow}$ $\phi(x+2\pi) = \phi(x) \;\forall x.$

put $x = 0$

$\phi(2\pi) = \phi(0)$

Also $\phi'(x+2\pi) = \phi'(x)$

(1 Mark) $\Rightarrow \phi'(0) = \phi'(2\pi).$

$\underset{\sim}{\Leftarrow}$. To prove $\phi(x+2\pi) = \phi(x) \;\forall x.$

Define $\chi(x) \equiv \phi(x+2\pi) - \phi(x)$

$\chi(0) = \phi(2\pi) - \phi(0)$

(2 Marks) $= 0$

$\chi'(0) = \phi'(2\pi) - \phi'(0)$

$= 0$

χ is a solution because both $\phi(x)$ & $\phi(x+2\pi)$ are solutions.

By the uniqueness of the I.V.P: (*)

(1 Mark) $x(x) \equiv 0$.

(*) : $x'' + (\sin x)\, x + x = 0$

$x(0) = 0$

$x'(0) = 0$

$\underline{x(x) \equiv 0} \Rightarrow\!\!\!\!\!\!/\ \underline{\phi(x + 2\pi) = \phi(x)}$.

2.

(a) $$v(x) = \frac{1}{x(\log x)^2} e^{-\int_1^x 0\, dt}$$

$$= \frac{1}{x(\log x)^2} e^{\text{Const}} \quad (1 \text{ Mark})$$

Take $$u'(x) = \frac{1}{x(\log x)^2} \quad (1 \text{ Mark})$$

$$\Rightarrow u(x) = \int_{x_0}^{x} \frac{(1/x)}{(\log x)^2} dx \quad (1 \text{ Mark})$$

put $\log x = t$

$(1/x) dx = dt$

$x = x_0 \Rightarrow t = \log x_0$

$x = x \Rightarrow t = \log x$

$$u(x) = \int_{\log x_0}^{\log x} \frac{1}{t^2} dt$$

$$= -1/t \Big|_{\log x_0}^{\log x}$$

$$= -\frac{1}{\log x} + \frac{1}{\log x_0} \quad (1 \text{ Mark})$$

$$\phi_2(x) = u(x)\cdot \phi_1(x)$$

$$= \left[-\frac{1}{\log x} + \frac{1}{\log x_0}\right] x^{1/2}\log x$$

$$= -x^{1/2} + x^{1/2}\log(x - x_0)$$

$$= -x^{1/2} + \boxed{\frac{x^{1/2}\log x}{\log x_0}} \to \phi_1(x).$$

So take $\phi_2(x) = x^{\frac{1}{2}}$ (1 Mark)

Answer $\phi_2(x) = x^{1/2}$ (1 Mark)

(b). $y'' - 4xy' - (2-4x^2)y = 0$

$$v(x) = \frac{1}{x^2 e^{2x^2}} e^{-\int_1^x -4t\,dt} \quad \text{(2 Mark)}$$

$$= \frac{1}{x^2 e^{2x^2}} e^{2x^2}\cdot e^{-2}. \quad \text{(1 Mark)}$$

Take $u'(x) = \frac{1}{x^2}$. (1 Mark)

$\Rightarrow u(x) = \frac{-1}{x}$. (1 Mark)

$$\phi_2(x) = -\frac{1}{x}\, x\, e^{x^2}$$

$$= -e^{x^2}. \quad \text{(2 Marks)}$$

3. Derivation of $\phi_1(x) = J_0(x) \longrightarrow$ (3 Marks) ③

$=$ " " "

Derivation of $\phi_2(x) = K_0(x) \longrightarrow$ (3 Marks)

All solutions of $x^2y'' + xy' + x^2y = 0$ are $c_1\phi_1(x) + c_2\phi_2(x)$.

$\hookrightarrow$ (1 Mark.

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