Question 3

(1)

$\leftarrow \phi(x)$, $\phi'(x)$ are continuous on $[-3,3]$.

$\phi(0) = 0, \quad \phi'(0) = 1.$

$y'' - 9y = 0 \quad$ on $[0,1]$

$\Rightarrow \phi_1(x) = c_1 e^{3x} + c_2 e^{-3x}.$

$$\left.\begin{aligned} 0 = \phi_1(0) &= c_1 + c_2 \\ 1 = \phi_1'(0) &= 3c_1 - 3c_2 \end{aligned}\right\} \Rightarrow \begin{aligned} 2c_1 &= \frac{1}{3} \\ c_1 &= \frac{1}{6} \end{aligned}$$

$$c_2 = -\frac{1}{6}.$$

(2 Marks) $\phi_1(x) = \frac{1}{6}\left[e^{3x} - e^{-3x}\right]$ on $[0,1]$

$y'' - 16y = 0 \quad$ on $[1,2]$

$\Rightarrow \phi_2(x) = c_3 e^{4x} + c_4 e^{-4x}$ on $[1,2]$

$\phi_1(1) = \phi_2(1)$

(1/2 Marks) $\leftarrow \frac{1}{6}\left[e^3 - e^{-3}\right] = c_3 e^4 + c_4 e^{-4} \rightarrow (1)$

$\phi_1'(1) = \phi_2'(1)$

(1/2 Marks) $\frac{1}{6}\left[3e^3 + 3e^{-3}\right] = 4c_3 e^4 - 4c_4 e^{-4}$

$\rightarrow (2)$

$$c_3 e^8 + c_4 = \frac{e^4}{6}\left[e^3 - e^{-3}\right]$$

$$c_3 e^8 - c_4 = \frac{e^4}{8}\left[e^3 + e^{-3}\right]$$

$$2c_3 e^8 = \frac{e^4}{2}\left\{\frac{e^3}{3} - \frac{e^{-3}}{3} + \frac{e^3}{4} + \frac{e^{-3}}{4}\right\}$$

$$c_3 = \frac{e^{-4}}{4}\left\{\frac{7e^3}{12} - \frac{e^{-3}}{12}\right\}$$

(1 Mark)

$$= \frac{7}{48e} - \frac{e^{-7}}{48}$$

$$c_4 = \frac{e^4}{6}\left[e^3 - e^{-3}\right] - \frac{e^8}{48}\left[\frac{7}{e} - e^{-7}\right]$$

(1 Mark)

$$= \frac{e^7}{6} - \frac{e}{6} - \frac{7}{48}e^7 + \frac{e}{48}$$

$$= \frac{e^7}{48} - \frac{7e}{48}$$

(1 Mark) $\phi_1(x) = \frac{1}{6}\left[e^{3x} - e^{-3x}\right]$ on $[0,1]$.

$$\phi_2(x) = \frac{1}{48}\left\{7e^{4x-1} - e^{4x-7} + e^{-4x+7} - 7e^{-4x+1}\right\}.$$

(2 Marks)