Equations with real coefficients

Suppose that the constants $a_1, \ldots, a_n$ in
$$\mathcal{L}(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y$$
are all real numbers. Then

$p(r) = r^n + a_1 r^{n-1} + \cdots + a_n$ has all real coefficients.

This implies that $\overline{p(r)} = p(\bar{r})$.

It follows now, if r_1 is a root of $p(r)$ then so is $\bar{r}_1$. Thus the roots of $p(r)$ whose imaginary parts do not vanish occur in conjugate pairs.

If there are s distinct roots of $p(r)$, let us enumerate them as follows:
$$r_1, \bar{r}_1, \; r_2, \bar{r}_2, \ldots, r_j, \bar{r}_j, \; r_{2j+1}, \ldots, r_s.$$
where $r_k = \sigma_k + i\tau_k$, $(k = 1, 2, \ldots, j;$

σ_k, τ_k real; $\tau_k \neq 0)$,

and $r_{2j+1}, \ldots, r_s$ are real.

Suppose r_k has multiplicity m_k then we have
$$2(m_1 + m_2 + \cdots + m_j) + m_{2j+1} + \cdots + m_s = n.$$

Corresponding to these roots we have the n linearly independent solutions:

$$\left.\begin{array}{l} e^{r_1 x},\ x e^{r_1 x},\ \dots,\ x^{m_1-1} e^{r_1 x}; \\ e^{\bar{r}_1 x},\ x e^{\bar{r}_1 x},\ \dots,\ x^{m_1-1} e^{\bar{r}_1 x};\ \dots; \\ e^{r_s x},\ x e^{r_s x},\ \dots,\ x^{m_s-1} e^{r_s x}, \end{array}\right\} \longrightarrow (*)$$

of $\mathcal{L}(y) = 0$.

Every solution is a linear combination, with constant coefficients, of these functions.

Instead of those functions we consider the following n linearly independent functions:

Write $x^h e^{r_k x} = x^h e^{(\sigma_k + i\tau_k)x}$

$$= x^h e^{\sigma_k x}[\cos\tau_k x + i \sin\tau_k x].$$

$$x^h e^{\bar{r}_k x} = x^h e^{(\sigma_k - i\tau_k)x}$$

$$= x^h e^{\sigma_k x}[\cos\tau_k x - i \sin\tau_k x].$$

for $1 \le k \le j,\ 0 \le h \le m_k - 1$.

Notice that $\mathcal{L}(x^h e^{\sigma_k x}\cos\tau_k x) = 0$

because $L.H.S = \mathcal{L}\left[\frac{1}{2}x^h(e^{r_k x} + e^{\bar{r}_k x})\right]$

$$= 0.$$

Similarly $\mathcal{L}(x^h e^{\sigma_k x} \sin \tau_k x) = 0.$

Thus every solution is a linear combination with constant cofficients of the following n functions:

$$(**) \quad \left\{ \begin{array}{l} e^{\sigma_1 x} \cos \tau_1 x, \; x e^{\sigma_1 x} \cos \tau_1 x, \ldots, x^{m_1 - 1} e^{\sigma_1 x} \cos \tau_1 x; \\ e^{\sigma_1 x} \sin \tau_1 x, \; x e^{\sigma_1 x} \sin \tau_1 x, \ldots, x^{m_1 - 1} e^{\sigma_1 x} \sin \tau_1 x; \\ \vdots \\ e^{r_s x}, \; x e^{r_s x}, \ldots, x^{m_s - 1} e^{r_s x}. \end{array} \right.$$

These are all real-valued and they are linearly independent.

<u>Proof of linearly independence of this new set.</u>

Consider the linear combination equal to zero.

Just consider any two terms of these:

$$x^h e^{\sigma_k x} \cos \tau_k x, \quad x^h e^{\sigma_k x} \sin \tau_k x.$$

$$c\, x^h e^{\sigma_k x} \cos \tau_k x + d\, x^h e^{\sigma_k x} \sin \tau_k x$$

where c & d are constants.

Using $x^h e^{\sigma_k x} \cos \tau_k x$

$$= \frac{1}{2} x^h \left[e^{r_k x} + e^{\bar{r}_k x} \right]$$

and $x^h e^{\sigma_k x} \sin \tau_k x$

$$= \frac{1}{2i} x^h \left[e^{r_k x} - e^{\bar{r}_k x} \right]$$

We find that we have a linear combination of the functions in $\circledast$ equal to zero, and the terms involving $x^h e^{r_k x}$, $x^h e^{\bar{r}_k x}$ will be

$$c x^h e^{\sigma_k x} \cos \tau_k x + d x^h e^{\sigma_k x} \sin \tau_k x$$

$$= c x^h \left[\frac{e^{r_k x} + e^{\bar{r}_k x}}{2} \right] + d x^h \frac{\left[e^{r_k x} - e^{\bar{r}_k x} \right]}{2i}$$

$$= c x^h \left[\frac{e^{r_k x} + e^{\bar{r}_k x}}{2} \right] + i d x^h \left[\frac{-e^{r_k x} + e^{\bar{r}_k x}}{2} \right]$$

$$= \frac{(c - i d)}{2} x^h e^{r_k x} + \frac{(c + i d)}{2} x^h e^{\bar{r}_k x}.$$

Since the functions in $\circledast$ involving $x^h e^{r_k x}$ & $x^h e^{\bar{r}_k x}$ are linearly Independent

$$c+id=0 \quad \& \quad c-id=0$$

$$\Rightarrow c=0 \quad \& \quad d=0.$$

Thus the solutions given in ** are linearly independent.

If ϕ is any real-valued solution of $\mathcal{L}(y)=0$, then ϕ is a linear combination of the real solutions given in ** with real coefficients.

Proof of the above statement:

Let us denote the solutions in ** as $\phi_1, \phi_2, \ldots, \phi_n$.

Now $\phi = c_1\phi_1 + \cdots + c_n\phi_n$ for some constants $c_1, \ldots, c_n$.

We have to show that $\mathrm{Im}(c_i)=0 \ \forall i = 1$ to n. $(i=1,2,\ldots,n)$.

Since $\phi, \phi_1, \ldots, \phi_n$ are all real valued, we have

$$0 = \mathrm{Im}(\phi) = \mathrm{Im}(c_1)\phi_1 + \cdots + \mathrm{Im}(c_n)\phi_n$$

Since $\phi_1, \phi_2, \ldots, \phi_n$ are linearly independent we must have

$$\mathrm{Im}(c_1) = 0, \quad \mathrm{Im}(c_2) = 0, \ldots, \mathrm{Im}(c_n) = 0.$$

This shows that the coefficients are all real numbers.

— X —

If ϕ is a solution of $\mathcal{L}(y) = 0$ which is such that

$$\phi(x_0) = \alpha_1$$
$$\phi'(x_0) = \alpha_2$$
$$\vdots$$
$$\phi^{(n-1)}(x_0) = \alpha_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are <u>real</u> constants, then <u>ϕ is real valued</u>.

<u>Proof of the above statement:</u>

We have to show that $\mathrm{Im}(\phi) = 0$.

If $\phi = u + i\psi$

$\bar{\phi} = u - i\psi$

then $\mathrm{Im}(\phi) = \psi = \dfrac{\phi - \bar{\phi}}{2i}$.

~~Let us say~~ $\psi \equiv \dfrac{\phi - \overline{\phi}}{2i}$.

Is ψ a solution of $\mathcal{L}(y) = 0$?

YES. because

$$\mathcal{L}(\phi) = 0 \;, \; \& \quad \overline{\mathcal{L}(\phi)} = 0$$

$$\Rightarrow \mathcal{L}(\overline{\phi}) = 0$$

(Because $a_1, \ldots, a_n$ are real).

NOW $\phi(x_0) = \alpha_1$, $\phi'(x_0) = \alpha_2$, $\ldots$, $\phi^{(n-1)}(x_0) = \alpha_n$

$$\Rightarrow \overline{\phi}(x_0) = \alpha_1, \; \overline{\phi'}(x_0) = \alpha_2, \ldots, \overline{\phi^{(n-1)}}(x_0) = \alpha_n$$

$$\Rightarrow \overline{\phi}(x_0) = \alpha_1, \; \overline{\phi}'(x_0) = \alpha_2, \ldots, \overline{\phi}^{(n-1)}(x_0) = \alpha_n$$

$$\psi(x_0) = \frac{1}{2i}\left[\phi(x_0) - \overline{\phi}(x_0)\right] = \frac{1}{2i}\left[\alpha_1 - \alpha_1\right] = 0$$

$$\vdots$$

$$\psi^{(n-1)}(x_0) = \frac{1}{2i}\left[\phi^{(n-1)}(x_0) - \overline{\phi}^{(n-1)}(x_0)\right] =$$

$$\frac{1}{2i}\left[\alpha_n - \alpha_n\right] = 0.$$

By the uniqueness theorem we have

$$\psi(x) \equiv 0.$$

$$\Rightarrow \operatorname{Im}(\phi) = 0.$$

We summarize the previous results in a theorem:

Theorem 19: Suppose the constants $a_1, a_2, \ldots, a_n$ in the equation

$$\mathcal{L}(y) \equiv y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$$

are all real. ① There exists a set of n linearly independent real-valued solutions:

$$e^{\sigma_1 x}\cos\tau_1 x,\ x e^{\sigma_1 x}\cos\tau_1 x, \ldots, x^{m_1-1} e^{\sigma_1 x}\cos\tau_1 x,$$

$$e^{\sigma_1 x}\sin\tau_1 x,\ x e^{\sigma_1 x}\sin\tau_1 x, \ldots, x^{m_1-1} e^{\sigma_1 x}\sin\tau_1 x;$$

$$\vdots$$

$$e^{r_s x},\ x e^{r_s x}, \ldots, x^{m_s-1} e^{r_s x}.$$

② Every real-valued solution is a linear combination of these functions (solutions) with real coefficients.

③ If a solution satisfies real initial conditions, it is real-valued.

The importance of Theorem 19 is that in many practical problems differential equations are encountered with real coefficients and the real solutions are the ones sought.

For example: Consider the equation
$$y^{(4)} + y = 0,$$
which arises in the study of the deflection of beams. The characteristic polynomial is given by $p(r) = r^4 + 1$, and its roots are

$$\frac{1}{\sqrt{2}}(1+i), \quad \frac{1}{\sqrt{2}}(1-i), \quad \frac{1}{\sqrt{2}}(-1+i), \quad \frac{1}{\sqrt{2}}(-1-i).$$

Thus every real solution ϕ of $y^{(4)} + y = 0$ has the form:

$$\phi(x) = e^{\frac{x}{\sqrt{2}}}\left[c_1 \cos\left(\frac{x}{\sqrt{2}}\right) + c_2 \sin\left(\frac{x}{\sqrt{2}}\right)\right] + e^{-\frac{x}{\sqrt{2}}}\left[c_3 \cos\left(\frac{x}{\sqrt{2}}\right) + c_4 \sin\left(\frac{x}{\sqrt{2}}\right)\right]$$

where c_1, c_2, c_3, c_4 are real constants.

How to obtain the roots of any polynomial $p(z) = z^n - c$, with c, a complex constant?

$$p(z) = 0 \Rightarrow z^n = c$$

$$\Rightarrow r^n e^{in\theta} = |c| e^{i\alpha}$$

$$\Rightarrow r = |c|^{1/n}.$$

$$\& \quad n\theta + 2k\pi = \alpha$$

for $k = 0, 1, 2, \dots n-1$.

or $\theta = \dfrac{\alpha - 2k\pi}{n}$, $k = 0, 1, 2, \dots, n-1$.

Therefore $z_1, z_2, \dots, z_n$ are the roots of the polynomial $p(z) = z^n - c$

with $z_{k+1} = |c|^{1/n}\left[\cos\left(\dfrac{\alpha + 2k\pi}{n}\right) + i \sin\left(\dfrac{\alpha + 2\pi k}{n}\right)\right]$

$(k = 0, 1, 2, \dots, n-1)$.

In the previous example $p(z) = z^4 + 1$,
$c = -1 = e^{i\pi}$ ie $\alpha = \pi$.

$$z_1 = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}(1+i).$$

$$z_2 = \cos\left(\frac{\pi+2\pi}{4}\right) + i\sin\left(\frac{\pi+2\pi}{4}\right)$$

$$= \cos\left(\frac{\pi}{4}+\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{2}\right)$$

$$= -\sin\left(\frac{\pi}{4}\right) + i\cos\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}(-1+i)$$

$$z_3 = \cos\left(\frac{\pi+4\pi}{4}\right) + i\sin\left(\frac{\pi+4\pi}{4}\right)$$

$$= \cos\left(\frac{\pi}{4}+\pi\right) + i\sin\left(\frac{\pi}{4}+\pi\right)$$

$$= -\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right).$$

$$= +\frac{1}{\sqrt{2}}(-1-i)$$

$$z_4 = \cos\left(\frac{\pi+6\pi}{4}\right) + i\sin\left(\frac{\pi+6\pi}{4}\right)$$

$$= \cos\left(\frac{\pi}{4}+\frac{3\pi}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{3\pi}{2}\right)$$

$$= \cos\left(\frac{\pi}{4}\right) + i \cdot -\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}(1-i).$$

The nonhomogeneous equation of order n :

Let $b(x)$ be a continuous function on an interval I, and consider the equation :

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b(x),$$

where $a_1, a_2, \ldots, a_n$ are constants.

If ψ_p is a particular solution of $L(y) = b(x)$ and ψ is any solution, then

$$L(\psi - \psi_p) = L(\psi) - L(\psi_p) = b - b = 0.$$

Thus $\psi - \psi_p$ is a solution of the homogeneous equation $L(y) = 0$ and this implies that any solution ψ of $L(y) = b(x)$ can be written in the form

$$\psi = \psi_p + c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n,$$

with $\phi_1, \phi_2, \dots, \phi_n$ are 'n' linearly independent solutions of the homogeneous equation and ψ_p is any particular solution of $\mathcal{L}(y) = b(x)$. $c_1, c_2, \dots c_n$ are constants.

To find a particular solution ψ_p we proceed just as in the case of $n = 2$, ie we use the variation of constants method. ie

To find n functions $u_1, u_2, \dots, u_n$ so that

$$\psi_p = u_1 \phi_1 + \dots + u_n \phi_n \text{ is}$$

a solution.

We need 'n' equations to solve for 'n' unknowns $u_1, u_2, \dots, u_n$.

First equation is $u_1' \phi_1 + \dots + u_n' \phi_n = 0$.

Then $\psi_p' = u_1 \phi_1' + \dots + u_n \phi_n'$.

If $u_1' \phi_1' + \dots + u_n' \phi_n' = 0$ we have

$$\psi_p'' = u_1 \phi_1'' + \dots + u_n \phi_n''.$$

Thus if $u_1', \ldots, u_n'$ satisfy

$$\left.\begin{aligned}
u_1'\phi_1 + \cdots + u_n'\phi_n &= 0\\
u_1'\phi_1' + \cdots + u_n'\phi_n' &= 0\\
&\vdots\\
u_1'\phi_1^{(n-2)} + \cdots + u_n'\phi_n^{(n-2)} &= 0\\
u_1'\phi_1^{(n-1)} + \cdots + u_n'\phi_n^{(n-1)} &= b
\end{aligned}\right\} \quad (1)$$

and we see that

$$\begin{aligned}
\psi_p &= u_1\phi_1 + \cdots + u_n\phi_n\\
\psi_p' &= u_1\phi_1' + \cdots + u_n\phi_n'\\
&\vdots\\
\psi_p^{(n-1)} &= u_1\phi_1^{(n-1)} + \cdots + u_n\phi_n^{(n-1)}\\
\psi_p^{(n)} &= u_1\phi_1^{(n)} + \cdots + u_n\phi_n^{(n)} + b
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{L}(\psi_p) &= u_1\mathcal{L}(\phi_1) + \cdots + u_n\mathcal{L}(\phi_n) + b\\
&= b
\end{aligned}$$

The whole problem is now reduced to solving the linear system (1) in the above.

The determinant of the coefficient is just $W(\phi_1, \phi_2, \dots, \phi_n)$. which is never zero when $\phi_1, \phi_2, \dots \phi_n$ are linearly independent solutions of $\mathcal{L}(y) = 0$.

Therefore there are unique functions $u_1', u_2', \dots, u_n'$ satisfying ① and they are given by:

$$u_k'(x) = \frac{W_k(x)\, b(x)}{W(\phi_1, \dots, \phi_n)(x)} \quad (k = 1, \dots n.)$$

where $W_k(x)$ is the determinant obtained from $W(\phi_1, \phi_2, \dots, \phi_n)$ by replacing the k-th column (ie $\phi_k, \phi_k', \dots, \phi_k^{(n-1)}$) by $\{0, 0, 0, \dots, 1$.

If x_0 is any point in I, we may take for u_k the function given by

$$u_k(x) = \int_{x_0}^{x} \frac{W_k(t)\, b(t)}{W(\phi_1, \dots, \phi_n)(t)}\, dt.$$

$(k = 1, 2, \dots, n)$.

The particular solution ψ_p now takes the form :

$$\psi_p(x) = \sum_{k=1}^{n} \phi_k(x) \int_{x_0}^{x} \frac{W_k(t)\, b(t)}{W(\phi_1, \cdots \phi_n)(t)}\, dt \quad (2)$$

<u>Theorem 20</u> : Let $b(x)$ be a continuous ~~function~~ function on an interval I, and let $\phi_1, \phi_2, \ldots, \phi_n$ be n linearly independent solutions of $L(y) = 0$ on I. Every solution ψ of $L(y) = b(x)$ can be written as

$$\psi = \psi_p + c_1 \phi_1 + \cdots + c_n \phi_n$$

where ψ_p is a particular solution of $L(y) = b(x)$, and $c_1, \ldots, c_n$ are constants. Every such ψ is a solution of $L(y) = b(x)$. A particular solution ψ_p is given by equation (2) in the above.

— x —

To prove that $\psi_p(x)$ given by equation 2 satisfies :

$$\psi_p(x_0) = 0$$

$$\psi_p'(x_0) = 0$$

$$\vdots$$

$$\psi_p^{(n-1)}(x_0) = 0 .$$

$$\psi_p(x_0) = \sum_{k=1}^{n} \phi_k(x) \int_{x_0}^{x_0} (\cdots) dt = 0 .$$

$$\psi_p'(x_0) = \phi_1'(x_0)\, u_1(x_0) + \phi_2'(x_0)\, u_2(x_0) + \cdots + \phi_n'(x_0)\, u_n(x_0)$$

$$= 0 .$$

$$\vdots$$

$$\psi_p^{(n-1)}(x_0) = 0 .$$

——— x ———

<u>Example</u>: Solve $y^{(3)} + y'' + y' + y = 1$.

which satisfy $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$.

The homogeneous eqn is

$$y^{(3)} + y'' + y' + y = 0.$$

and the characteristic polynomial is

$$p(r) \equiv r^3 + r^2 + r + 1.$$

The roots of $p(r)$ are $r = i, -i, -1$.

Since we are interested in a solution satisfying real initial conditions we take for independent solutions :

$$\phi_1(x) = \cos x, \quad \phi_2(x) = \sin x$$
$$\& \ \phi_3(x) = e^{-x}.$$

To obtain a particular solution of

$y^{(3)} + y'' + y' + y = 1$ of the form

$u_1\phi_1 + u_2\phi_2 + u_3\phi_3$ we must solve the following equations for $u_1', u_2' \ \& \ u_3'$:

$$u_1'\phi_1 + u_2'\phi_2 + u_3'\phi_3 = 0$$

$$u_1'\phi_1' + u_2'\phi_2' + u_3'\phi_3' = 0$$

$$u_1'\phi_1'' + u_2'\phi_2'' + u_3'\phi_3'' = 1$$

which in this case reduce to

$$(\cos x)\cdot u_1' + (\sin x)\, u_2' + e^{-x} u_3' = 0$$

$$(-\sin x)\, u_1' + (\cos x)\, u_2' - e^{-x} u_3' = 0$$

$$(-\cos x)\, u_1' - (\sin x)\, u_2' + e^{-x} u_3' = 1.$$

$$W(\phi_1, \phi_2, \phi_3)(x) = \begin{vmatrix} \cos x & \sin x & e^{-x} \\ -\sin x & \cos x & -e^{-x} \\ -\cos x & -\sin x & e^{-x} \end{vmatrix}$$

By using the formula for the Wronskian

$$W(\phi_1, \phi_2, \phi_3)(x) = e^{-x}\, W(\phi_1, \phi_2, \phi_3)(0)$$

(because $a_1 = 1$)

$$= e^{-x} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} = 2e^{-x}.$$

Solving for u_1', u_2' & u_3' we get

$$u_1'(x) = \frac{W_1(x)\, b(x)}{W(\phi_1, \phi_2, \phi_3)(x)}$$

$$= \begin{vmatrix} 0 & \sin x & e^{-x} \\ 0 & \cos x & -e^{-x} \\ 1 & -\sin x & e^{-x} \end{vmatrix} \times \frac{1}{2} e^{x}$$

$$= -\frac{1}{2}(\cos x + \sin x).$$

$$u_2'(x) = \frac{1}{2}(\cos x - \sin x)$$

$$u_3'(x) = \frac{1}{2} e^{x}.$$

Integrating we obtain

$$u_1(x) = \frac{1}{2}(\cos x - \sin x)$$

$$u_2(x) = \frac{1}{2}(\sin x + \cos x)$$

$$u_3(x) = \frac{1}{2} e^{x}.$$

Therefore a particular solution is given by

$$\psi_p(x) = u_1 \phi_1 + u_2 \phi_2 + u_3 \phi_3$$

$$= \frac{1}{2}(\cos x - \sin x)\cos x + \frac{1}{2}(\sin x + \cos x)\sin x + \frac{1}{2} e^{x} \cdot e^{-x} = 1.$$

The general solution is given by

$$\psi(x) = 1 + c_1 \cos x + c_2 \sin x + c_3 e^{-x}.$$

where c_1, c_2 & c_3 are constants.

We determine these constants using the initial conditions

$$\psi(0) = 0$$
$$\psi'(0) = 1$$
$$\psi''(0) = 0$$

$$0 = \psi(0) = 1 + c_1 + c_3 \Rightarrow c_1 + c_3 = -1$$
$$1 = \psi'(0) = -c_3 + c_2 \Rightarrow c_2 - c_3 = 1$$
$$0 = \psi''(0) = -c_1 + c_3 \Rightarrow c_1 = c_3.$$

$$c_1 = -\frac{1}{2}, \quad c_2 = \frac{1}{2}, \quad c_3 = -\frac{1}{2}.$$

Therefore the solution is given by

$$\psi(x) = 1 + \frac{1}{2}\left(\sin x - \cos x - e^{-x}\right)$$

—— x ——