Linear Equations with Variable Coefficients

A linear differential equation of order n with variable coefficients is an equation of the form:

$$a_0(x)\, y^{(n)} + a_1(x)\, y^{(n-1)} + \dots + a_n(x) y = b(x).$$

where $a_0, a_1, \dots, a_n$ are complex-valued functions on some real interval I.

For some points $x \in I$, $a_0(x) = 0$, such points are called singular points. We consider this special case in the next chapter.

Therefore in this chapter we assume that $a_0(x) \neq 0$ on I. (ie for all points $x \in I$, $a_0(x) \neq 0$.)

So we can safely divide by $a_0(x)$.

$$y^{(n)} + \frac{a_1(x)}{a_0(x)} y^{(n-1)} + \cdots + \frac{a_n(x)}{a_0(x)} y = \frac{b(x)}{a_0(x)}.$$

For simplicity we again consider the coefficients as $a_1(x), \ldots, a_n(x)$ and the RHS as $b(x)$:

$$y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_n(x) y = b(x).$$

In the operator form we write as

$$\mathcal{L}(y) = b(x).$$

If $b(x) \equiv 0 \ \forall x \in I$ we say

$\mathcal{L}(y) = 0$ is a homogeneous eqn.

If $b(x) \neq 0$ for some $x \in I$, the eqn

$\mathcal{L}(y) = b(x)$ is a Nonhomogeneous eqn.

<u>Definition of a solution to the linear equation of order 'n'</u>: We say a function 'ϕ' is a solution if ϕ is differentiable 'n' times on I and ϕ satisfies the equation $-\ \mathcal{L}(\phi) = b(x)$.

$$\mathcal{L} : C^n(I) \longrightarrow C(I).$$
$$\phi \longmapsto \mathcal{L}(\phi).$$

(Space of n times continuously differentiable functions.)

$$\mathcal{L}(\phi)(x) = \phi^{(n)}(x) + a_1(x)\,\phi^{(n-1)}(x) + \cdots + a_n(x)\,\phi(x).$$

In this chapter we assume that the complex-valued functions $a_1, \ldots, a_n, b$ are continuous on some real interval I.

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Most of the results we developed in the previous chapter for the case when $a_1, \ldots, a_n$ are constants continue to be valid in the more general case we are now considering.

The major difficulty with linear equations with variable coefficients, from a practical point of view, is that it is rare that we can solve the equations <u>in terms of elementary functions</u>, such as the exponential and trignometric functions. Thus there is no analogue of the rather powerful Theorem 11 of the previous chapter.

However, in case $a_1, \ldots, a_n, b$ have convergent power series expansions the solutions will have this property also, and these series solutions can be obtained by a simple formal process.

Although in many cases it is not possible to express a solution of

$$y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_n(x) y' = b(x)$$

in terms of elementary functions, it can be proved that solutions always exist.

Existence of solution of Initial value problem (~~IVP~~ IVP):

Theorem 1: Let $a_1(x), \ldots, a_n(x)$ be continuous functions on an interval I containing the point x_0. If $\alpha_1, \ldots, \alpha_n$ are any n constants, there exists a solution of

$$L(y) = y^{(n)} + a_1(x)\, y^{(n-1)} + \cdots + a_n(x)\, y = 0$$

on I satisfying

$$\phi(x_0) = \alpha_1,\ \phi'(x_0) = \alpha_2,\ \ldots,\ \phi^{(n-1)}(x_0) = \alpha_n$$

Now we assume this result and prove in the forth-coming chapter on system of equations.

Just as in the case where the coefficients a_j $(j=1,\ldots,n)$ are constants, the uniqueness of the solution ϕ given in Theorem 1 is demonstrated with the aid of an estimate for

$$\|\phi(x)\|^2 = |\phi(x)|^2 + |\phi'(x)|^2 + \cdots + |\phi^{(n-1)}(x)|^2$$

<u>Theorem 2</u>: Let $b_1, \ldots, b_n$ be non-negative constants such that for all x in I

$$|a_j(x)| \le b_j, \quad (j=1,\ldots,n),$$

and define k by

$$k = 1 + b_1 + \cdots + b_n.$$

If x_0 is a point in I, and ϕ is a solution of $L(y)=0$ on I, then

$$\|\phi(x_0)\| e^{-k|x-x_0|} \le \|\phi(x)\| \le \|\phi(x_0)\| e^{k|x-x_0|}$$

for all x in I.

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<u>Proof</u>: Since $\mathcal{L}(\phi) = 0$ we have

$$\phi^{(n)}(x) = -a_1(x)\,\phi^{(n-1)}(x) - \cdots - a_n(x)\phi(x),$$

and therefore

$$|\phi^{(n)}(x)| \le |a_1(x)|\,|\phi^{(n-1)}(x)| + \cdots + |a_n(x)|\,|\phi(x)|$$

$$\le b_1\,|\phi^{(n-1)}(x)| + \cdots + b_n\,|\phi(x)|.$$

The proof of Theorem 13, of the previous chapter, now applies if we substitute b_j, everywhere in place of $|a_j|$.

We remark that if I is a closed and bounded interval, that is of the form $a \le x \le b$, with a, b real, and if the a_j are continuous on I, then there always exist finite constants b_j such that $|a_j(x)| \le b_j$ on I.

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Uniqueness Theorem (Theorem 3):

Let x_0 be in I, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be any 'n' constants. There is at most one solution ϕ of $L(y) = 0$ on I satisfying:

$$(*) \longleftarrow \phi(x_0) = \alpha_1,\ \phi'(x_0) = \alpha_2, \ldots, \phi^{(n-1)}(x_0) = \alpha_n$$

Proof: Let ϕ, ψ be two solutions of $L(y) = 0$ on I satisfying the conditions $(*)$ at x_0, and consider

$\chi = \phi - \psi$. We wish to prove $\chi(x) \equiv 0$.

Even though the functions a_j are continuous on I they need not be bounded there. for eg: $a_1(x) = x$ is not bounded on $0 \le x < \infty$; and $a_1(x) = \frac{1}{x}$ is not bounded on $0 < x \le 1$.

Therefore we can not apply Theorem 2 directly. However, let x be any point on I other than x_0, and let J be any closed bounded interval in I

which contains x_0 and x. On this interval the functions a_j are bounded, that is

$$|a_j(x)| \leq b_j \, , \, (j=1,2,\ldots,n),$$

on J for some constants b_j, which may depend on J. Now we apply Theorem 2 to χ defined on J. We have $\mathcal{L}(\chi)=0$ on J, and $\|\chi(x_0)\|=0$. Therefore $\|\chi(x)\|=0$, and hence $\phi(x)=\psi(x)$. Since x was chosen to be any point in I other than x_0, we have proved $\phi(x)=\psi(x)$ for all x in I.

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If $\phi_1,\ldots,\phi_m$ are any m solutions of the n-th order equation $\mathcal{L}(y)=0$, on an interval I, and $c_1,\ldots,c_m$ are any m constants, then

$$\mathcal{L}(c_1\phi_1+\cdots+c_m\phi_m) = c_1\mathcal{L}(\phi_1)+\cdots + c_m\mathcal{L}(\phi_m),$$

which implies that $c_1\phi_1+\cdots+c_m\phi_m$ is also a solution.

As in the case of an L with constant coefficients, every solution of $L(y)=0$ is a linear combination of any n linearly independent solutions.

Using Theorem 1 we construct 'n' linearly independent solutions, and show that every solution is a linear combination of these. In the next section we show that every solution is a linear combination of any n linearly independent solutions.

<u>Theorem 4</u>: There exist 'n' linearly independent solutions of $L(y)=0$ on I.

<u>Proof</u>: Let x_0 be a point in I. According to Theorem 1 there is a solution ϕ_1 of $L(y)=0$ satisfying

$$\phi_1(x_0)=1,\ \phi_1'(x_0)=0,\ \ldots,\ \phi_1^{(n-1)}(x_0)=0.$$

In general for each $i=1,2,\ldots,n$

there is a solution ϕ_i satisfying
$$\phi_i^{(i-1)}(x_0)=1, \quad \phi_i^{(j-1)}(x_0)=0, \quad j \neq i.$$

The solutions $\phi_1, \dots, \phi_n$ are linearly independent on I, for suppose there are constants $c_1, c_2, \dots, c_n$ such that

$$(*) \leftarrow c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) = 0$$

for all x in I. Differentiating we see that

$$(**) \begin{cases} c_1\phi_1'(x) + c_2\phi_2'(x) + \cdots + c_n\phi_n'(x) = 0 \\ c_1\phi_1''(x) + c_2\phi_2''(x) + \cdots + c_n\phi_n''(x) = 0 \\ \vdots \\ c_1\phi_1^{(n-1)}(x) + c_2\phi_2^{(n-1)}(x) + \cdots + c_n\phi_n^{(n-1)}(x) = 0. \end{cases}$$

In particular, the equations $(*)$ & $(**)$ must hold at x_0. Putting $x = x_0$ in $(*)$ we find that $c_1 \cdot 1 + 0 + \cdots + 0 = 0$
$$\Rightarrow c_1 = 0.$$
Putting $x = x_0$ in the equations $(**)$ we obtain $c_2 = c_3 = \cdots = c_n = 0$.
and thus the solutions $\phi_1, \dots, \phi_n$ are linearly independent.

Theorem 5: Let $\phi_1, \ldots, \phi_n$ be the n solutions of $L(y)=0$ on I satisfying

$$\phi_i^{(i-1)}(x_0) = 1, \quad \phi_i^{(j-1)}(x_0) = 0, \quad j \neq i.$$

or

$$\phi_i^{(j-1)} = \delta_{ij} \text{ (kronecker delta)}.$$

If ϕ is any solution of $L(y)=0$ on I, there are n constants $c_1, c_2, \ldots, c_n$ such that

$$\phi = c_1\phi_1 + \cdots + c_n\phi_n.$$

Proof: Let

$$\phi(x_0) = \alpha_1, \ \phi'(x_0) = \alpha_2, \ldots, \phi^{(n-1)}(x_0) = \alpha_n$$

and consider the function

$$\psi = \alpha_1\phi_1 + \alpha_2\phi_2 + \cdots + \alpha_n\phi_n$$

It is a solution of $L(y)=0$ and clearly

$$\psi(x_0) = \alpha_1\phi_1(x_0) + \alpha_2\phi_2(x_0) + \cdots + \alpha_n\phi_n(x_0)$$

$$= \alpha_1,$$

|||ly we get $\psi'(x_0) = \alpha_2$

$$\vdots$$

$$\psi^{(n-1)}(x_0) = \alpha_n.$$

Thus ψ is a solution of $L(y)=0$ having the same initial conditions at x_0 as ϕ. By Theorem 3, we must have $\phi=\psi$, that is

$$\phi = \alpha_1\phi_1 + \alpha_2\phi_2 + \cdots + \alpha_n\phi_n$$

we have proved the theorem with the constants

$$c_1=\alpha_1,\ c_2=\alpha_2, \ldots, c_n=\alpha_n.$$

The Wronskian and linear independence

Theorem 6: If $\phi_1, \ldots, \phi_n$ are n solutions of $L(y)=0$ on I, they are linearly independent there if and only if

$$W(\phi_1, \ldots, \phi_n)(x) \neq 0 \quad \forall x \in I.$$