$$d_k = \sum_{j=0}^{k-1} \left[(j+r)\,\alpha_{k-j} + \beta_{k-j} \right] c_j \longrightarrow (*)$$

$$c_k = \frac{-d_k}{q(r+k)}, \quad k = 1, 2, 3, \ldots \longrightarrow (**)$$

Put $k=1$:

$$d_1 = (r\,\alpha_1 + \beta_1)\, c_0$$

Since $c_0 = 1$, we have

$$d_1 \equiv D_1(r) = r\,\alpha_1 + \beta_1 .$$

and

$$c_1 = \frac{-d_1}{q(r+1)} .$$

$$= -\frac{D_1(r)}{q(r+1)}$$

put $k=2$:

$$D_2(r) \equiv d_2 = (r\,\alpha_2 + \beta_2)\, c_0 + \left[(r+1)\,\alpha_1 + \beta_1 \right] c_1$$

$$c_2 = \frac{-d_2}{q(r+2)} = \frac{-D_2(r)}{q(r+2)} .$$

So for computing the Coefficients $c_1, c_2, \ldots$

we require that

$$q(r+1) \neq 0, \; q(r+2) \neq 0$$

$$\ldots\ldots \; q(r+k) \neq 0$$

ie $q(r_1 + k) \neq 0$

Since $q(r)$ has only two roots

$$r_1 + k \neq r_2$$

or $r_1 - r_2 \neq k$

That is why we have to consider the case $r_1 - r_2 = k$ as an exceptional case. (ie difference of two roots r_1 & r_2 is a positive integer k).

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The case of repeated roots $r_1 = r_2$ is also considered as an exceptional case.

Theorem: Consider the equation

$$x^2 y'' + a(x)\, x y' + b(x)\, y = 0$$

where a, b have convergent power series expansions for $|x| < r_0$, $r_0 > 0$.

Let r_1, r_2 ($\mathrm{Re}\, r_1 \geq \mathrm{Re}(r_2)$) be the roots of the indicial polynomial

$$q(r) = r(\cancel{R}r-1) + a(0)\, r + b(0).$$

For $0 < |x| < r_0$ there is a solution ϕ_1 of the form $\phi_1(x) = |x|^{r_1} \sum_0^\infty c_k x^k \ (c_0 = 1)$ where the series converges for $|x| < r_0$.

If $r_1 - r_2$ is not zero or a positive integer, there is a second solution ϕ_2 for $0 < |x| < r_0$ of the ~~two~~ form $\phi_2(x) = |x|^{r_2} \sum \tilde{c}_k x^k$, $(c_0 = 1)$,

where the series converges for
$|x| < r_0$.

The coefficients c_k, $\tilde{c}_k$ can be obtained by substitution of the solutions into the differential equation.