

Lecture 4

11/1/16

①

In the previous three lectures we have seen the methods of Characteristics to solve various types of first order PDEs (linear, semilinear and Quasilinear)

We also discussed the method of Lagrange to find the General solutions of any Quasilinear PDE.

Now the question is whether general solutions cover all solutions?

In other words, do All solutions are of the form of general solution?

The answer is Need NOT BE.

Recall the situation in the
ODEs: Consider $y' = \sqrt{y}$

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By the method of separation of variables we can solve to find a general solution of the above ODE.

$$\frac{dy}{dx} = \sqrt{y}$$

or

$$\int \frac{dy}{\sqrt{y}} = \int dx$$

$$\Rightarrow 2\sqrt{y} = x + C$$

$$\text{or } y = \frac{1}{4}(x+C)^2$$

For any arbitrary $C \in \mathbb{C}$ we have general solutions of the above form.

But notice that $y=0$ (the trivial solution)

is not included in this. (3)

i.e. No particular value of c will produce the trivial solution.

In fact when we solved $y' = \sqrt{y}$ we divided by $\sqrt{y}$ assuming $y \neq 0$.

Now consider the PDE situation:

$$u_x + u_y = \sqrt{u}.$$

Let us find the general solution.

Step 1: Find $\phi(x, y, z) = \text{const.}$

For that solve:

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{\sqrt{z}}.$$

Consider first and last expressions and integrate to find

$$2\sqrt{z} = x + C_1$$

$$\text{or } z = \frac{1}{4}(x + C_1)^2.$$

(4)

Take $\phi(x, y, z) \equiv 2\sqrt{z} - x = c_1$.

Step 2: Find $\psi(x, y, z) = \text{const}$

For that consider first two expressions and integrate to find

$$x - y = c_2.$$

Take $\psi(x, y, z) = x - y = c_2$.

Step 3: Write the general solution as $\phi = f(\psi)$, for some arbitrary function f .

$$\text{i.e. } 2\sqrt{z} - x = f(x - y).$$

or $z \equiv u(x, y)$

the general solution is $= \frac{1}{4} (x + f(x - y))^2$

Observe that the trivial solution $u(x, y) = 0$ is not covered by the general solution.

⑤

So we conclude that the method of Lagrange fails to produce ALL solutions of PDE.

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Now we will demonstrate that the method of characteristics also cannot solve every problem:

Already we have seen a couple of limitations of the method of characteristics:

- ① Regarding the problem of solving for s and t in terms of x and y , the inverse function theorem tells us that this can be achieved provided the Jacobian matrix is noningular:

$$J \equiv \det \begin{pmatrix} x_s & y_s \\ x_t & y_t \end{pmatrix} \equiv x_s y_t - y_s x_t \neq 0.$$

(6)

Here J is a function of s & t .

$$J \equiv x_s(s, t) \cdot y_t(s, t) - y_s(s, t) \cdot x_t(s, t)$$

In particular at $t = 0$,

$$\begin{aligned} J &= x_s(s, 0) \cdot y_t(s, 0) - y_s(s, 0) \cdot x_t(s, 0) \\ &= f'(s) \cdot b(f(s), g(s)) \\ &\quad - g'(s) \cdot a(f(s), g(s)) \end{aligned}$$

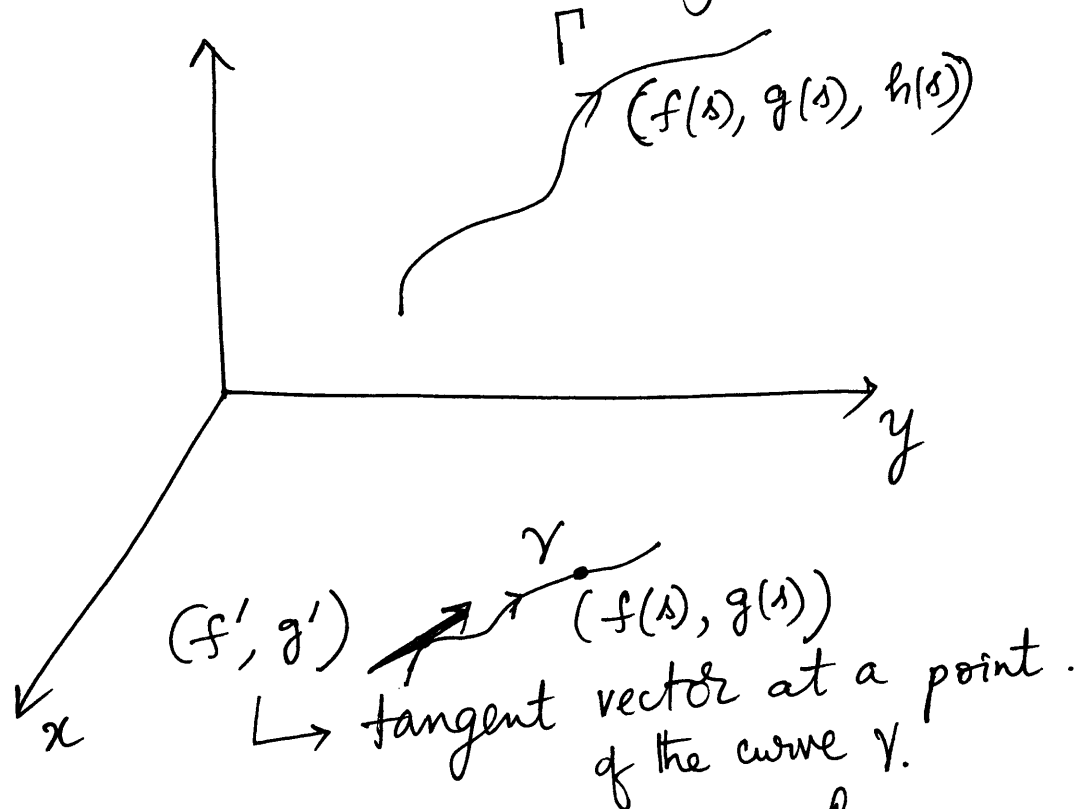
So $J \neq 0$ at $t = 0 \Rightarrow$

$$\frac{f'(s)}{g'(s)} \neq \frac{a}{b} \quad \text{or} \quad \frac{f'(s)}{a} \neq \frac{g'(s)}{b}$$

ie $(f'(s), g'(s))$ is nowhere parallel to (a, b) .

Geometrically, the projection of Γ into the xy -plane is a curve $\gamma = \{f(s), g(s)\}$ that is nowhere

parallel to the vector field (a, b) . (7)



The characteristic curves are ^{always} tangent to the given vector field (a, b, c) .

So the conclusion is that the method of characteristics works only for the initial data Γ which is NONCHARACTERISTIC.

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(8)

Limitation (2) of the method of characteristics:

Away from γ (ie for larger values of t) the solution may develop a singularity where $J=0$.

Even if $J \neq 0$ for all values of s and t , the solution may develop a blow up type of singularity if the equation for $\frac{dz}{dt}$ is nonlinear.

Example: $u_x + 2u_y = u^2$, $u(x,0) = h(s)$

$$J = x_s y_t - y_s x_t \text{ at } t=0.$$

$$= 1 \cdot 2 - 0 \cdot 1 = 2 \neq 0.$$

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2, \quad \frac{dz}{dt} = z^2$$

$$\begin{aligned} & \Downarrow \\ & \left. \begin{aligned} x(s,t) &= t + c_1(s) \\ s = x(s,0) &= c_1(s) \end{aligned} \right\} \Rightarrow \begin{aligned} x(s,t) &= t + s. \\ x_s &= 1, \quad x_t = 1 \end{aligned} \end{aligned}$$

$$\frac{dy}{dt} = 2 \Rightarrow y(s, t) = 2t + c_2(s) \quad (9)$$

$$0 = y(s, 0) = c_2(s)$$

$$y(s, t) = 2t$$

$$y_t = 2, \quad y_s = 0.$$

$$J = x_s y_t - y_s x_t$$

$$= 1 \cdot 2 - 0 \cdot 1 = 2 \neq 0.$$

for all values of s and t .

But the solution,

$$u(x, y) = \frac{h(x - y/2)}{1 - \frac{y}{2} h(x - \frac{y}{2})}$$

$u(x, y)$ blows up (becomes infinite) if y becomes large enough to cause the denominator to vanish.

So the conclusion is that we need methods to find the solutions which are singular (jump discontinuities) at some points (x, y) .

(10)

③ Third limitation of the method of characteristics:

The characteristic curves may intersect at some points.

Example: (Inviscid Burgers equation)

$$u_t + u u_x = 0, \quad u(x, 0) = h(x).$$

$$\frac{dx}{z} = \frac{dy}{1} = \frac{dz}{0}.$$

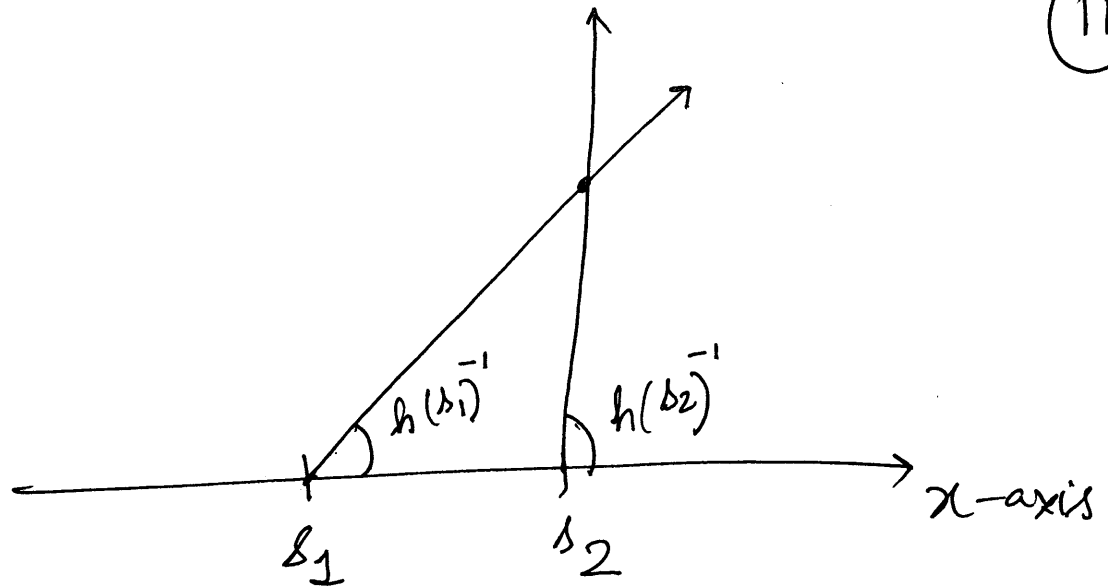
$$\text{or } \frac{dx}{dt} = z, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = 0$$

$$\begin{array}{l} \Downarrow \qquad \qquad \qquad \Downarrow \qquad \qquad \qquad \Downarrow \\ \left. \begin{array}{l} x(s, t) = h(s)t + c_3(s) \\ s = x(s, 0) = c_3(s) \end{array} \right\} \Rightarrow y(s, t) = t \quad \left| \quad \begin{array}{l} z = c_2(s) \\ = h(s). \end{array} \right. \\ \hline \boxed{\therefore x(s, t) = h(s)t + s} \end{array}$$

or
For fixed s_1 , $x(s_1, t) = h(s_1) \cdot y + s_1$.

Slope of the straight line is $h(s_1)$.
 z has the constant value $h(s_1)$ along this straight line.

(11)



If s_2 is another point s_2 such that $s_1 < s_2$ but $h(s_1) > h(s_2)$, then the projected characteristic curves beginning from s_1 and s_2 will intersect at a point (t, y) where

$$y = t = \frac{s_2 - s_1}{h(s_1) - h(s_2)} > 0.$$

$$x = h(s_1)t + s_1 = h(s_2)t + s_2$$

$$\Rightarrow [h(s_1) - h(s_2)]t = s_2 - s_1.$$

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