

Exercises

(1)

1. Let $\underline{f}$ be the vector-valued function defined on $R: |x| \leq 1, |\underline{y}| \leq 1, \underline{y} \in \mathbb{C}^2$ by

$$\underline{f}(x, \underline{y}) = (y_2^2 + 1, x + y_1^2)$$

(a) Find an upper bound M for $|\underline{f}(x, \underline{y})|$ for $(x, \underline{y}) \in R$.

(b) Compute a Lipschitz constant K for $\underline{f}$ on R .

Solution:

$$(a) \quad |\underline{f}(x, \underline{y})| = |(y_2^2 + 1, x + y_1^2)|$$

$$= |y_2^2 + 1| + |x + y_1^2|$$

$$\leq 1 + |y_2|^2 + |x| + |y_1|^2$$

$$= 1 + |x| + |y_1|^2 + |y_2|^2$$

$$= 1 + |x| + |\underline{y}|^2 \leq 3 = M$$

$$(|\underline{y}|^2 = \underline{y} \cdot \underline{y} = y_1^2 + y_2^2).$$

(2)

$$\left| \frac{\partial f}{\partial y_1} \right| = |(0, 2y_1)| = 2|y_1| \leq 2$$

$$\left| \frac{\partial f}{\partial y_2} \right| = |(2y_2, 0)| = 2|y_2| \leq 2.$$

$$K = 2.$$

2. Consider the system of two equations:

$$\begin{aligned} y_1' &= ay_1 + by_2 \\ y_2' &= cy_1 + dy_2 \end{aligned}$$

where a, b, c, d are constants.

(a) If this system is written in the form $\underline{y}' = \underline{f}(x, \underline{y})$, what is $\underline{f}$?

(b) Show that the $\underline{f}$ of (a) satisfies a Lipschitz condition for all $(x, \underline{y})$

where $x \in \mathbb{R}$ and $\underline{y} \in \mathbb{C}^2$.

(c) Show that the $\underline{f}$ of (a) is linear in $\underline{y}$, i.e. $\underline{f}(x, \alpha \underline{y} + \beta \underline{z}) = \alpha \underline{f}(x, \underline{y}) + \beta \underline{f}(x, \underline{z})$

for all real x , complex constants α, β and all $\underline{y}, \underline{z} \in \mathbb{C}^2$. (3)

Solution:

$$(a) \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{pmatrix} = \underline{f}(x, \underline{y}).$$

$$(b) \left| \frac{\partial \underline{f}}{\partial y_1} \right| = |(a, c)| = |a| + |c|$$

$$\left| \frac{\partial \underline{f}}{\partial y_2} \right| = |(b, d)| = |b| + |d|$$

Lipschitz constant $K = \max \{ |a| + |c|, |b| + |d| \}$

$$(c) \underline{f}(x, \alpha \underline{y} + \beta \underline{z}) = \begin{pmatrix} a(\alpha y_1 + \beta z_1) + b(\alpha y_2 + \beta z_2) \\ c(\alpha y_1 + \beta z_1) + d(\alpha y_2 + \beta z_2) \end{pmatrix}$$

$$= \alpha \begin{pmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{pmatrix} + \beta \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix}$$

$$= \underline{\alpha \underline{f}(x, \underline{y}) + \beta \underline{f}(x, \underline{z})}$$

(4)

3. Find a solution $\underline{\Phi}$ of the system $y_1' = y_1$
 $y_2' = y_1 + y_2$ which satisfies

$$\underline{\Phi}(0) = (1, 2).$$

Solution: $y_1' = y_1 \Rightarrow y_1(x) = ce^x$
 $1 = y_1(0) = c \cdot e^0 = c \cdot 1$
 $\Rightarrow c = 1$

$$y_2' = e^x + y_2 \Rightarrow y_2' - y_2 = e^x$$

Homogeneous solution: $y_2' - y_2 = 0 \Rightarrow y_2 = e^x$

P.I. = $y_2(x) = Ae^x + Bxe^x$

$$= (A+Bx)e^x$$

$$y_2'(x) = (A+Bx)e^x + Be^x$$

$$y_2'(x) - y_2(x) = Be^x = e^x$$

$$\Rightarrow B = 1$$

$$y_2(x) = e^x + (A+x)e^x$$

$$= (A+x+1)e^x$$

$$2 = y_2(0) = (A+1) \Rightarrow A = 1$$

$$y_2(x) = e^x + (1+x)e^x = (x+2)e^x$$

Exercises in Page 253.

Consider the IVP : $y_1' = y_2^2 + 1$

$$y_2' = y_1^2.$$

$$y_1(0) = 0, \quad y_2(0) = 0.$$

①

(a) If this problem is denoted by

$$\underline{y}' = \underline{f}(x, \underline{y}), \quad \underline{y}(0) = \underline{y}_0.$$

What are $\underline{f}$ and $\underline{y}_0$?

(b) Show that the $\underline{f}$ of (a) satisfies the conditions of continuity and Lipschitz continuity on $R : |x| \leq 1, |\underline{y}| \leq 1$.

Compute a bound M , a Lipschitz

constant K and an α .

(c) Compute the first three successive approximations $\underline{\Phi}_0, \underline{\Phi}_1, \underline{\Phi}_2$.

Solution: (a) $\underline{f}(x, \underline{y}) = \begin{pmatrix} y_2^2 + 1 \\ y_1^2 \end{pmatrix}$

$\underline{y}_0 = (0, 0).$

(2)

$$\textcircled{b} \quad |f(x, y)|$$

$$= |y_2^2 + 1| + |y_1^2|$$

$$\leq 1 + |y_1^2| + |y_2^2|$$

$$= 1 + |y|^2 \leq 2 = M.$$

$$\left| \frac{\partial f}{\partial y_1} \right| = |(0, 2y_1)| = 2|y_1| \leq 2$$

$$\left| \frac{\partial f}{\partial y_2} \right| = |(2y_2, 0)| = 2|y_2| \leq 2.$$

$$K = 2.$$

$$\alpha = \min \left\{ a, \frac{b}{M} \right\} = \min \left\{ 1, \frac{1}{2} \right\} = \frac{1}{2}.$$

$$\textcircled{c} \quad \underline{\Phi}_0 = (0, 0).$$

$$\underline{\Phi}_1(x) = \left(\int_0^x (y^2 + 1) dt, \int_0^x 0 dt \right)$$

$$= (x, 0)$$

$$\begin{aligned}\Phi_2(x) &= \left(\int_0^x (t^2 + 1) dt, \int_0^x t^2 dt \right) \quad (3) \\ &= \left(\cancel{\frac{x^3}{3} + x}, \cancel{\frac{x^3}{3}} \right) \\ &= \left(x, \frac{x^3}{3} \right).\end{aligned}$$

Consider the system

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \begin{pmatrix} 3y_1 + xy_3 \\ y_2 + x^2y_3 \\ 2xy_1 - y_2 + e^x y_3 \end{pmatrix}$$

Show that every IVP for this system has a unique solution which exists for all real x .

$$\frac{\partial f}{\partial y_1} = (3, 0, 2x)$$

$$\frac{\partial f}{\partial y_2} = (0, 1, -1)$$

(4)

$$\frac{\partial f}{\partial y_3} = (x, x^2, e^x)$$

For nonlocal existence of the solution of the IVP we need Lipschitz continuity on any "strip"

$$S: |x| \leq a, |y| < \infty.$$

$$\left| \frac{\partial f}{\partial y_1} \right| = |(3, 0, 2x)| = 3 + 2|x| \leq 3 + 2a$$

$$\left| \frac{\partial f}{\partial y_2} \right| = |(0, 1, -1)| = 1 + 1 = 2.$$

$$\left| \frac{\partial f}{\partial y_3} \right| = |(x, x^2, e^x)| = |x| + |x|^2 + |e^x| \leq a + a^2 + e^a.$$

This shows that f satisfies the Lipschitz condition on any "strip" S .

