

Greens functions and Poisson Kernel ①

First note that $w(\underline{y}) \equiv \frac{1}{|\underline{x}-\underline{y}|}$ is a harmonic function in $\mathbb{R}^3$, for some fixed $\underline{x} \in \Omega \subset \mathbb{R}^3$ (a bounded domain Ω).

Because Laplace's equation is translation invariant, the above is true.

Consider the translation operator:

$$\tilde{\underline{x}} = \underline{x} + \underline{a}, \text{ for a fixed } \underline{a}.$$

It is easy to see that $\Delta_{\tilde{\underline{x}}} = \Delta_{\underline{x}}$.

Already we have seen that $\frac{1}{|\underline{x}|}$ or $\frac{1}{|\underline{y}|}$ is a harmonic function in $\mathbb{R}^3$.

Now for a fixed $\underline{x} \in \mathbb{R}^3$

$\frac{1}{|\underline{x}-\underline{y}|}$ is just a translation

of $\frac{1}{|\underline{y}|}$ by a fixed $\underline{x}$.

So $\frac{1}{|\underline{x}-\underline{y}|} \equiv w(\underline{y})$ is a harmonic fn.

Representation Theorem in 3 dimensions (2)

Let Ω be bounded domain in $\mathbb{R}^3$ and let $\hat{n}$ be the unit exterior normal to $\partial\Omega$. Let $u \in C^2(\bar{\Omega})$.

Then the value of u at any point $x \in \Omega$ is given by the formula

$$u(x) = \frac{1}{4\pi} \int_{\partial\Omega} \left[\frac{1}{|x-y|} \cdot \frac{\partial u(y)}{\partial \hat{n}} - u(y) \frac{\partial}{\partial \hat{n}} \frac{1}{|x-y|} \right] ds \\ - \frac{1}{4\pi} \int_{\Omega} \frac{\Delta u(y)}{|x-y|} dy.$$

Proof: Consider the Green's identity

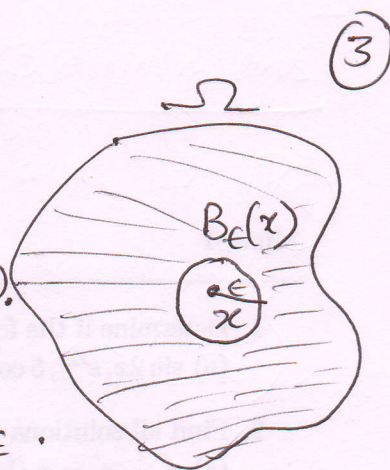
$$\int_{\partial\Omega} \left(u \frac{\partial v}{\partial \hat{n}} - v \frac{\partial u}{\partial \hat{n}} \right) ds = \int_{\Omega} [u \Delta v - v \Delta u] dy.$$

Now take $v(y) = \frac{1}{|x-y|}$ which is a harmonic function and singular at $x \in \Omega$.

In order to be able to apply Green's identity, we consider a new domain $\Omega_\epsilon \equiv \Omega \setminus B_\epsilon(x)$.

Now apply Green's identity
on the domain Ω_ϵ .

Note that $\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(x)$.



We have $\Delta v = 0$ in Ω_ϵ .

In the RHS of the green's identity we
have only one term $-\int_{\Omega_\epsilon} \frac{\Delta u(y)}{|x-y|} dy$.

$$\begin{aligned} \text{LHS} &= \int_{\partial\Omega} \left[u(y) \frac{\partial}{\partial \hat{n}} \frac{1}{|x-y|} - \frac{1}{|x-y|} \frac{\partial u}{\partial \hat{n}}(y) \right] ds \\ &\quad + \int_{\partial B_\epsilon(x)} \left[u(y) \frac{\partial}{\partial \hat{n}} \frac{1}{|x-y|} - \frac{1}{|x-y|} \frac{\partial u}{\partial \hat{n}}(y) \right] ds \end{aligned}$$

Now let $\epsilon \rightarrow 0$ both sides.

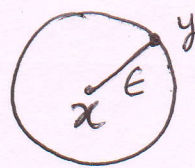
The first integral in the LHS does not
depend on ϵ .

The second integral in the LHS :

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\partial B_\epsilon(x)} \left[u(y) \frac{\partial}{\partial \hat{n}} \frac{1}{|x-y|} - \frac{1}{|x-y|} \frac{\partial u}{\partial \hat{n}}(y) \right] ds \\ = 4\pi \cdot u(x). \end{aligned}$$

On the boundary $\partial B_\epsilon(x)$

$$\frac{1}{|x-y|} = \frac{1}{\epsilon}$$



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$$\frac{\partial}{\partial \hat{n}} \frac{1}{|x-y|} = \frac{1}{\epsilon^2} \quad \left(\text{Inward direction will be } -\frac{1}{\epsilon^2} \right)$$

$$\int_{\partial B_\epsilon(x)} \left[u(y) \cdot \frac{\partial}{\partial \hat{n}} \frac{1}{|x-y|} - \frac{1}{|x-y|} \cdot \frac{\partial u}{\partial \hat{n}}(y) \right] ds$$

$$= \int_{\partial B_\epsilon(x)} \left[\frac{1}{\epsilon^2} u(y) - \frac{1}{\epsilon} \frac{\partial u}{\partial \hat{n}}(y) \right] ds$$

$$= \int_{\partial B_\epsilon(x)} \frac{1}{\epsilon^2} u(x) ds + \int_{\partial B_\epsilon(x)} \frac{1}{\epsilon^2} [u(y) - u(x)] ds - \int_{\partial B_\epsilon(x)} \frac{1}{\epsilon} \frac{\partial u}{\partial \hat{n}}(y) ds$$

$$= 4\pi u(x) + \int_{\partial B_\epsilon(x)} \left\{ \frac{1}{\epsilon^2} [u(y) - u(x)] - \frac{1}{\epsilon} \frac{\partial u}{\partial \hat{n}}(y) \right\} ds$$

Now as $\epsilon \rightarrow 0$ the above integral tends to zero:

$$\left| \int_{\partial B_\epsilon(x)} \left[\frac{1}{\epsilon^2} [u(y) - u(x)] - \frac{1}{\epsilon} \frac{\partial u}{\partial \hat{n}}(y) \right] ds \right|$$

$$\leq 4\pi \underbrace{\max_{y \in \partial B_\epsilon(x)} |u(y) - u(x)|}_{\rightarrow 0 \text{ because continuous in } \Omega} + 4\pi \epsilon \underbrace{\max_{y \in \bar{\Omega}} \|\nabla u(y)\|}_{\text{finite}}$$

$\rightarrow 0$ because continuous in Ω .

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Since $\frac{1}{|x-y|}$ is integrable at $x=y$

$$\lim_{\epsilon \rightarrow 0} \left[- \int_{\Omega_\epsilon} \frac{\Delta u(y)}{|x-y|} dy \right] = - \int_{\Omega} \frac{\Delta u(y)}{|x-y|} dy.$$

Hence the theorem.

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Representation Theorem in 2 dimensions:

Let Ω be bounded domain in $\mathbb{R}^2$ and let $\hat{n}$ be the unit exterior normal to $\partial\Omega$.

Let $u \in C^2(\bar{\Omega})$. Then the value of u at any point $x \in \Omega$ is given by the formula:

$$u(x) = \frac{1}{2\pi} \int_{\Omega} \Delta u(y) \log|x-y| dy$$

$$+ \frac{1}{2\pi} \int_{\partial\Omega} \left[u(y) \frac{\partial}{\partial n} \log|x-y| - \log|x-y| \frac{\partial u(y)}{\partial n} \right] ds.$$

Proof: Because Laplace equation is translation invariant $\log|x-y|$ is harmonic function in $\mathbb{R}^2$ for some fixed $x \in \mathbb{R}^2$.

The rest of the proof goes in the similar way as we proved in 3 dimensions.

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